When do the harmonic Hardy spaces with distinct indices coincide on a hyperbolic Riemann surface?

Dedicated to Professor Hisashi Ishida on his sixtieth birthday

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Abstract

Let $R$ be a hyperbolic Riemann surface. Suppose that $1 \leq p < q \leq \infty$. In this paper we give a characterization that two harmonic Hardy spaces $h_p(R)$ and $h_q(R)$ coincide with each other by using the term of the Martin boundary $\Delta^M$ of $R$. Let $\Delta^M$ be the minimal Martin boundary of $R$. In the case that $p > 1$ it holds that $h_p(R)$ coincides with $h_q(R)$ if and only if there exists a nullset $N$ of $\Delta^M$ with respect to the harmonic measure such that $\Delta^M \setminus N$ consists of finitely many points with positive harmonic measure. In the case that $p = 1$ it holds that $h_1(R)$ coincides with $h_q(R)$ if and only if $\Delta^M$ consists of finitely many points with positive harmonic measure.

Keywords: hyperbolic Riemann surface, harmonic Hardy space, Martin boundary, minimal Martin boundary, harmonic measure

1. Introduction

Denote by $O_G$ the class of open Riemann surfaces $R$ such that there exist no Green’s functions on $R$. We say that an open Riemann surface $R$ is parabolic (resp. hyperbolic) if $R$ belongs (resp. does not belong) to $O_G$.

For an open Riemann surface $R$, we denote by $HP_+(R)$ and $HB_+(R)$ the classes of non-negative harmonic functions and non-negative bounded harmonic functions on $R$, respectively. Denote by $MHB_+(R)$ the class of all finite limit functions of monotone increasing sequences of $HB_+(R)$. Set $HX(R) = HX_+(R) - HX_-(R)$ ($X = P, B$), where $HX_+(R) - HX_-(R) = \{ h_1 - h_2 \mid h_j \in HX_+(R) \ (j = 1, 2) \}$, and $MHB(R) = MHB_+(R) - MHB_-(R)$. Then, $HB(R)$ are the class of bounded harmonic functions on $R$. $MHB(R)$ is called the class of quasi-bounded functions on $R$. It is well-known that if $R$ is parabolic, then $HX(R)$ ($X = P, B$) and $MHB(R)$ consist of constant functions (cf. [4]).

Hereafter, we consider only hyperbolic Riemann surfaces $R$. Let $\Delta^M = \Delta^{R,M}$ and $\Delta^I = \Delta^{R,M}_I$ the Martin boundary of $R$ and the minimal Martin boundary on $R$, respectively. We refer to [1] for the details about the Martin boundary. Denote by $h_p(R)$ ($1 \leq p \leq \infty$) the harmonic Hardy space with index $p$ on $R$ (see Definition of harmonic Hardy space in the next section). It is
well-known that, if \( 1 \leq p < q \), \( h_q(R) \subset h_p(R) \). It is natural to ask when the converse inclusion relation holds. The purpose of this paper is to answer the question.

**Theorem 1.** Suppose that \( R \) is hyperbolic and \( 1 < p < q \leq \infty \). Then the followings are equivalent:

(i) \( h_p(R) = h_q(R) \),

(ii) there exists a nullset \( N \) of \( \Delta^M \) with respect to the harmonic measure such that \( \Delta^M_1 \setminus N \) consists of finitely many points with positive harmonic measure,

(iii) \( \dim h_p(R) = \dim h_q(R) < \infty \),

where \( \dim h_p(R) \) is the dimension of the linear space \( h_p(R) \).

**Theorem 2.** Suppose that \( R \) is hyperbolic, \( p = 1 \) and \( 1 < q \leq \infty \). Then the followings are equivalent:

(i) \( h_1(R) = h_q(R) \),

(ii) \( \Delta^M_1 \) consists of finitely many points with positive harmonic measure,

(iii) \( \dim h_1(R) = \dim h_q(R) < \infty \),

where \( \dim h_1(R) \) is the dimension of the linear space \( h_1(R) \).

As an immediate consequence of Theorem 2, by the fact that \( h_1(R) = HP(R) \) and definition of \( h_\infty(R) \), we obtain the following.

**Corollary** (cf. [2, Theorem]). Suppose that \( R \) is hyperbolic. Then the followings are equivalent:

(i) \( HP(R) = HB(R) \),

(ii) \( \Delta^M_1 \) consists of finitely many points with positive harmonic measure,

(iii) \( \dim HP(R) = \dim HB(R) < \infty \),

where \( \dim HP(R) \) is the dimension of the linear space \( HP(R) \).

## 2. Preliminaries

In this section we state several propositions in order to prove theorems in §1 in the next section. Denote by \( \omega^M_\zeta(\cdot) \) the harmonic measure on \( \Delta^M \) with respect to \( z \in R \). We also denote by \( k_\zeta(z) ((\zeta, z) \in (R \cup \Delta^M) \times R) \) the Martin kernel on \( R \) with pole at \( \zeta \). The following proposition plays fundamental role in the proof of Theorem 2.

**Proposition 1** (cf. [1, Hilfssatz 13.3]). Let \( \zeta \) belong to \( \Delta^M_1 \). Then the Martin kernel \( k_\zeta(\cdot) \) with pole at \( \zeta \) is bounded on \( R \) if and only if the harmonic measure \( \omega((\zeta)) \) of the singleton \( \{\zeta\} \) is positive.

The next proposition follows from the Martin representation theorem which is the most fundamental theorem in the Martin theory.

**Proposition 2.** Let \( u \) be an element in \( HP(R) \). There exists a signed measure \( \mu \) on \( \Delta^M \) such that \( \mu(\Delta^M \setminus \Delta^M_1) = 0 \) and \( u = \int_{\Delta^M_1} k_\zeta d\mu(\zeta) \).

**Proof of Proposition 2.** Let \( u \) be an element in \( HP(R) \). By definition of \( HP(R) \) there exist elements \( u_1 \) and \( u_2 \) of \( HP_+(R) \) with \( u = u_1 - u_2 \) on \( R \). By the Martin representation theorem(cf. [1, Satz 13.1]) we find the Radon measure \( \mu_j (j = 1, 2) \) on \( \Delta^M \) such that \( \mu_j(\Delta^M \setminus \Delta^M_1) = 0 (j = 1, 2) \).
and \( u_j = \int_{\Delta_{1}^h} k_\xi d\mu_j(\xi) (j = 1, 2) \). Set \( \mu = \mu_1 - \mu_2 \). Then \( \mu \) is a signed measure on \( \Delta^M \). We have
\[
\mu(\Delta^M \setminus \Delta_1^M) = \mu_1(\Delta^M \setminus \Delta_1^M) - \mu_2(\Delta^M \setminus \Delta_1^M) = 0 - 0 = 0
\]
and
\[
u = u_1 - u_2 = \int_{\Delta_{1}^h} k_\xi d\mu_1(\xi) - \int_{\Delta_{1}^h} k_\xi d\mu_2(\xi) = \int_{\Delta_{1}^h} k_\xi d\mu(\xi).
\]
We have the desired result.

**Definition** (cf. [3, Definition in p.437 and Theorem 4]). Let \( p \geq 1 \). Set
\[
h_p(R) = \begin{cases} \{u \mid u \text{ is harmonic on } R \text{ and } |u|^p \text{ has a harmonic majorant on } R\}, & \text{for } p \geq 1, \\ HB(R), & \text{for } p = \infty. \end{cases}
\]

We call \( h_p(R) \) the **harmonic Hardy space** with index \( p \) on \( R \). We remark that \( h_1(R) = HP(R) \) and that, if \( 1 \leq p < q \leq \infty \), \( h_p(R) \subset h_q(R) \).

**Proposition 3** (cf. [3, Definition in p.437 and Theorems 4 and 6]). Let \( p \) be a real number with \( 1 < p < \infty \). Fix a point \( z_0 \) of \( R \). The next conditions are equivalent.

(i) \( u \in h_p(R) \),

(ii) \( u \) has the minimal fine limit \( u^*(\zeta) \) at almost every point \( \zeta(\in \Delta^M) \) with respect to the harmonic measure \( \omega_\zeta^M \) such that \( u(z) = \int_{\Delta^M} u^*(\zeta)d\omega_\zeta^M(\zeta) \), and \( \int_{\Delta^M} |u^*(\zeta)|^pd\omega_\zeta^M(\zeta) < \infty \).

Set \( h_{p^*}(R) := h_p(R) \cap HP_{p^*}(R) \). By the above proposition it is easily seen that \( h_p(R) = h_{p^*}(R) - h_{p^*}(R) \) and \( h_{p^*}(R) \subset MHB_{p^*}(R) \).

### 3. Proof of Theorems

#### 3.1 Proof of Theorem 1

First we consider \( q \neq \infty \). Let \( p \) and \( q \) be real numbers with \( 1 < p < q \). Suppose that (i) holds.

Fix a point \( z_0 \) of \( R \). Further we suppose that there exists a point \( \zeta(\in \Delta^M) \) such that, for any positive \( \rho \), \( \omega_\zeta^M(U_{\rho}(\zeta)) > 0 \) and \( \omega_\zeta^M(|\zeta|) = 0 \), where \( U_{\rho}(\zeta) \) is the disc with center \( \zeta \) and radius \( \rho \) with respect to the standard metric on \( R \cup \Delta^M \). Hence, there exists a monotone decreasing sequence \( \{\rho_n\} \) with \( \lim_{n \to \infty} \rho_n = 0 \), \( \omega_0^M(U_{\rho_n}(\zeta)) > 0 \), \( \omega_0^M(U_{\rho_{n+1}}(\zeta)) \leq \rho_{n+1}/(q-p) \) \( (n \in \mathbb{N}) \). Set
\[
f^*(\xi) = \begin{cases} [\omega_0^M(U_{\rho_n}(\xi)) \setminus U_{\rho_{n+1}}(\xi))]^{-1/q}, & \text{for } \xi(\in U_{\rho_n}(\xi) \setminus U_{\rho_{n+1}}(\xi)), \\ 0, & \text{for } \xi(\in \Delta^M \setminus U_{\rho_n}(\xi)). \end{cases}
\]

And set \( f(z) = \int_{\Delta^M} f^*(\xi)d\omega_\zeta^M(\xi) \). Then, we find that \( f \) has a minimal fine limit \( f^*(\zeta) \) at almost every point \( \zeta(\in \Delta_1^M) \) with respect to \( \omega_\zeta^M \). Hence, we have
\[
\int_{\Delta^M} f^*(\xi)d\omega_\zeta^M(\xi) = \sum_{n=1}^{\infty} [\omega_0^M(U_{\rho_n}(\xi)) \setminus U_{\rho_{n+1}}(\xi))]^{-1}\omega_\zeta^M(U_{\rho_n}(\xi)) \setminus U_{\rho_{n+1}}(\xi))
\]

\[
= \sum_{n=1}^{\infty} 1 = \infty.
\]
and
\[
\int_{\Delta^1} f^*(\xi)^p d\omega^M_{\zeta}(\xi) = \sum_{n=1}^{\infty} \left[ \omega^M_{\zeta}(U_{\rho_n}(\xi) \setminus U_{\rho_{n+1}}(\xi)) \right]^{-p/q} \omega^M_{\zeta}(U_{\rho_n}(\xi) \setminus U_{\rho_{n+1}}(\xi)) \\
= \sum_{n=1}^{\infty} \left[ \omega^M_{\zeta}(U_{\rho_n}(\xi) \setminus U_{\rho_{n+1}}(\xi)) \right]^{1-p/q} \\
\leq \sum_{n=1}^{\infty} \left[ \omega^M_{\zeta}(U_{\rho_n}(\xi) \setminus U_{\rho_{n+1}}(\xi)) \right]^{(q-p)/q} \\
\leq \sum_{n=1}^{\infty} 1/n^2 < \infty.
\]

By Proposition 3 we find that \( f \in h_p(R) \setminus h_q(R) \). By (i) this is a contradiction. Hence, if \( \zeta \in \Delta^M \) satisfies that, for any positive \( \rho \), \( \omega^M_{\zeta}(U_{\rho}(\xi)) > 0 \). By the above fact, it holds that there exists a null set \( N \) of \( \Delta^M \) with respect to \( \omega^M_{\zeta} \) such that \( \Delta^1 \setminus N \) consists of at most countably many points with positive harmonic measure. To see this set

\[ N = \{ \xi \in \Delta^M \mid \text{there exists a positive } \rho_\xi \text{ with } \omega^M_{\zeta}(U_{\rho_\xi}(\xi)) = 0 \} \]

and set \( F = \Delta^M \setminus N \). Clearly \( F \cup N = \Delta^M \), \( F \cap N = \emptyset \) and, for any \( \xi \in F \), \( \omega^M_{\zeta}(\{\xi\}) > 0 \). Hence \( F \) is an at most countable subset of \( \Delta^1 \) because \( \omega^M_{\zeta}(\Delta^M \setminus \Delta^1) = 0 \). Hence it is sufficient to prove that \( \omega^M_{\zeta}(N) = 0 \). Set \( O = \cup_{\xi \in N} U_{\rho_{\xi}}(\xi) \). Clearly \( O \) is an open subset of \( R \cup \Delta^M \) and \( O \cap \Delta^M = N \). By the Lindelöf theorem there exists a sequence \( \{\xi_n\}_{n=1}^{\infty} \) of \( N \) with \( O = \cup_{n=1}^{\infty} U_{\rho_{\xi_n}}(\xi_n) \). Hence \( \omega^M_{\zeta}(O) \leq \sum_{n=1}^{\infty} \omega^M_{\zeta}(U_{\rho_{\xi_n}}(\xi_n)) = 0 \), and hence, \( \omega^M_{\zeta}(N) = 0 \).

Suppose that \#(\Delta^1 \setminus N) = \( N_0 \), where \#(\Delta^1 \setminus N) is the cardinal number of \( \Delta^1 \setminus N \). Set \( \Delta^1 \setminus N = \{\xi_n\}_{n=1}^{\infty} \). Hence there exists a subsequence \( \{\xi_{n_l}\}_{l=1}^{\infty} \) of \( \{\xi_n\} \) with \( \omega^M_{\zeta}(\{\xi_{n_l}\}) \leq 1/l^{2q/(q-p)} \) \( (l \in \mathbb{N}) \).

Set
\[
g^*(\xi) = \begin{cases} 
\frac{1}{\omega_{\zeta}(\{\xi_{n_l}\})}^{-1/q}, & \text{for } \xi = \xi_{n_l}, \\
0, & \text{for } \xi \in \Delta^M \setminus \{\xi_{n_l}\}_{l=1}^{\infty}.
\end{cases}
\]

And set \( g(z) = \int_{\Delta^M} g^*(\xi)d\omega^M_{\zeta}(\xi) \). Then, we find that \( g \) has a minimal fine limit \( g^*(\xi) \) at almost every point \( \xi \in \Delta^M \) with respect to \( \omega^M_{\zeta} \). Hence, we have
\[
\int_{\Delta^M} g^*(\xi)^qd\omega^M_{\zeta}(\xi) = \sum_{l=1}^{\infty} \frac{1}{\omega_{\zeta}(\{\xi_{n_l}\})}^{-1} \omega^M_{\zeta}(\{\xi_{n_l}\}) \\
= \sum_{n=1}^{\infty} 1 = \infty.
\]
and

\[
\int_{\Delta_M^1} g^*(\xi)^p d\omega_{M_20}(\xi) = \sum_{l=1}^{\infty} \left[ \omega_{M_20}(\{\xi_n\}) \right]^{1-p/q} \omega_{M_20}((\xi_n)) \\
= \sum_{l=1}^{\infty} [\omega_{M_20}(\{\xi_n\})]^{1-\frac{1}{p/q}} \\
\leq \sum_{l=1}^{\infty} [\omega_{M_20}(\{\xi_n\})]^{(q-p)/q} \\
\leq \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty.
\]

By Proposition 3 we find that \( g \in h_p(R) \setminus h_q(R) \). By (i) this is a contradiction. Hence, there exists a nullset \( N \) of \( \Delta_M^1 \) with respect to \( \omega_{M_20} \) such that \( \Delta_M^1 \setminus N \) consists of finitely many points with positive harmonic measure, and so, we get (ii).

Suppose that (ii) holds. Fix a point \( z_0 \) of \( R \). We can find a nullset \( N \) of \( \Delta_M^1 \) with respect to \( \omega_{M_20} \) such that \( \Delta_M^1 \setminus N \) consists of finitely many points with positive harmonic measure. Let \( n_0 \) be the cardinal number of \( \Delta_M^1 \setminus N \). Set \( \Delta_M^1 \setminus N = \{\xi_n\}_{n=1}^{n_0} \). Let \( p \) and \( q \) be real numbers with \( p < q \) and \( p > 1 \). Clearly \( h_q(R) \subset h_p(R) \). Take any element of \( h \) of \( h_p(R) \). By definition of \( h_p(R) \) \( h \) has a minimal fine limit \( h^*(\xi) \) at almost every point \( \xi \in \Delta_M^1 \) with respect to \( \omega_{M_20} \) such that \( h(z) = \int_{\Delta_M^1} h^*(\xi)d\omega_{M_20}(\xi) \) and \( \int_{\Delta_M^1} |h^*(\xi)|^q d\omega_{M_20}(\xi) < \infty \). By (ii) we have

\[
h(z) = \int_{\Delta_M^1} h^*(\xi)d\omega_{M_20}(\xi) = \sum_{n=1}^{n_0} h^*(\xi_n)\omega_{M_20}(\{\xi_n\})
\]

and

\[
\int_{\Delta_M^1} |h^*(\xi)|^q d\omega_{M_20}(\xi) = \sum_{n=1}^{n_0} |h^*(\xi_n)|^q \omega_{M_20}(\{\xi_n\}) < \infty.
\]

Hence \( h_p(R) \leq n_0 \) and \( |h^*(\xi_n)| < \infty \) \( (n = 1, \ldots, n_0) \). Thus,

\[
\int_{\Delta_M^1} |h^*(\xi)|^q d\omega_{M_20}(\xi) = \sum_{n=1}^{n_0} |h^*(\xi_n)|^q \omega_{M_20}(\{\xi_n\}) < \infty,
\]

and hence \( h \in h_q(R) \), that is, \( h_p(R) \subset h_q(R) \). Hence \( h_p(R) = h_q(R) \). Hence dim \( h_q(R) = \) dim \( h_p(R) \leq n_0 < \infty \). We get (iii).

Suppose that (iii) holds. Let \( p \) and \( q \) be real numbers with \( p < q \) and \( p > 1 \). Since \( h_p(R) \) and \( h_q(R) \) are linear spaces and \( h_q(R) \) is a subspace of \( h_p(R) \), by (iii), we find that \( h_p(R) = h_q(R) \). Hence we get (i).

Next we consider \( q = \infty \). Suppose that (i) holds. Take a real number \( p' \) with \( p' > p \). Then \( HB(R) \subset h_p(R) \subset h_p(R) \). By (i) \( h_p(R) = h_p(R) \). By the implication: (i) \( \Rightarrow \) (ii) in the case that \( q \neq \infty \) we get (ii).

Suppose that (ii) holds. Fix a point \( z_0 \) of \( R \). We can find a nullset \( N \) of \( \Delta_M^1 \) with respect to \( \omega_{M_20} \) such that \( \Delta_M^1 \setminus N \) consists of finitely many points with positive harmonic measure. Let \( n_0 \) be the cardinal number of \( \Delta_M^1 \setminus N \). Set \( \Delta_M^1 \setminus N = \{\xi_n\}_{n=1}^{n_0} \). Clearly \( HB(R) \subset h_p(R) \). Take any element \( h \) of
exists a positive constant \( \beta \).

Hence \( \dim h_p(R) \leq n_0 \) and \( |h^*(\xi_n)| < \infty \) \( (n = 1, \ldots, n_0) \). Thus, \( h \in HB(R) \), that is, \( h_p(R) \subseteq HB(R) \). Hence \( h_p(R) = HB(R) \). Hence \( \dim HB(R) = \dim h_p(R) \leq n_0 < \infty \). We get (iii).

Suppose that (iii) holds. Since \( h_p(R) \) and \( HB(R) \) are linear spaces and \( HB(R) \) is a subspace of \( h_p(R) \), by (iii), we find that \( h_p(R) = HB(R) \). Hence we get (i).

Therefore we have the desired result.

### 3.2 Proof of Theorem 2

First we consider \( q \neq \infty \). Suppose that (i) holds, that is, \( HP(R) = h_q(R) \) \( (q > 1) \). Let \( h \) be a minimal harmonic function on \( R \). Clearly \( h \in HP_q(R) \). By (i) \( h \in HP_q(R) \cap h_q(R) = h_{q+}(R) \). Since \( h_{q+}(R) \subseteq MB_{q+}(R) \), \( h \in MB_{q+}(R) \). Thus there exists a monotone increasing sequence \( \{h_n\}_{n=1}^{\infty} \) of \( MB_{q+}(R) \) such that \( h_n \neq 0 \) \( (n \in \mathbb{N}) \) and \( \lim_{n \to \infty} h_n = h \) on \( R \). By minimality of \( h \) there exists a positive constant \( \alpha \) such that \( h = \alpha h_1 \) on \( R \). Hence \( h \) is bounded on \( R \). Let \( \xi_h \) be the element of \( \Delta_1^M \) coresponding to \( h \). Fix a point \( z_0 \) of \( R \). Since \( h \) is minimal, there exists a positive constant \( \beta \) with \( h = \beta k_{\xi_h} \). Hence, because \( h \) is bounded on \( R \), by Proposition 1 we find that the harmonic measure \( \omega_{z_0}^M(\{\xi_h\}) \) of \( \{\xi_h\} \) is positive. Hence, \( \Delta_1^M \) consists of at most countably many points with positive harmonic measure.

Suppose that \( \#\Delta_1^M = \infty \). Set \( \Delta_1^M = \{\xi_n\}_{n=1}^{\infty} \). Hence there exists a subsequence \( \{\xi_n\}_{n=1}^{\infty} \) of \( \{\xi_n\}_{n=1}^{\infty} \) with \( \omega_{z_0}^M(\{\xi_n\}) \leq 1/\rho^q/(q-1) \) \( (l \in \mathbb{N}) \). Set

\[
\begin{align*}
    g^*(\xi) &= \begin{cases} 
    \omega_{z_0}^M(\{\xi_n\})^{-1/q}, & \text{for } \xi = \xi_n, \\
    0, & \text{for } \xi \in \Delta_1^M \setminus \{\xi_n\}_{n=1}^{\infty}.
\end{cases}
\end{align*}
\]

And set \( g(z) = \int g^*(\xi)d\omega_{z_0}^M(\xi) \). Then we have

\[
\int g^*(\xi)d\omega_{z_0}^M(\xi) = \sum_{n=1}^{\infty} \omega_{z_0}^M(\{\xi_n\})^{-1} \omega_{z_0}^M(\{\xi_n\}) = \sum_{n=1}^{\infty} 1 = \infty,
\]

By (ii) we have

\[
h(z) = \int_{\Delta_1^M} h^*(\xi)d\omega_{z_0}^M(\xi) = \sum_{n=1}^{n_0} h^*(\xi_n)\omega_{z_0}^M(\{\xi_n\})
\]

and

\[
\int_{\Delta_1^M} |h^*(\xi)|^p d\omega_{z_0}^M(\xi) = \sum_{n=1}^{n_0} |h^*(\xi_n)|^p \omega_{z_0}^M(\{\xi_n\}) < \infty.
\]

By definition of \( h_p(R) \) we find that \( h(z) = \sum_{n=1}^{n_0} h^*(\xi_n)\omega_{z_0}^M(\{\xi_n\}) \).
and
\[
\int g^*(\xi) d\omega^M_\infty(\xi) = \sum_{l=1}^\infty [\omega^M_\infty(\{\xi_n\})]^{-1/q} \omega^M_\infty(\zeta_n)
\]
\[
= \sum_{l=1}^\infty [\omega^M_\infty(\{\xi_n\})]^{1-1/q}
\]
\[
\leq \sum_{l=1}^\infty [\omega^M_\infty(\{\xi_n\})]^{(q-1)/q}
\]
\[
\leq \sum_{l=1}^\infty 1/l^2 < \infty.
\]
By Proposition 3 we find that \(g \in MHB(R) \setminus h_q(R)\). Since \(HP(R) \supset MHB(R)\), by (i), this is a contradiction. Hence \(\Delta^M_1\) consists of finitely many points with positive harmonic measure, and so we get (ii).

Suppose that (ii) holds. Let \(n_0\) be the cardinal number of \(\Delta^M_1\). Set \(\Delta^M_1 = \{\zeta_n\}_{n=1}^{n_0}\). Let \(q > 1\). Clearly \(h_q(R) \subset HP(R)\). Take any element \(h\) of \(HP(R)\). By Proposition 2 we find a signed measure \(\mu\) on \(\Delta^M\) such that \(\mu(\Delta^M \setminus \Delta^M_1) = 0\) and \(h(z) = \int_{\Delta^M_1} \omega^M_z(\{\xi\})d\mu(\xi)\). By (ii) we have
\[
h(z) = \int_{\Delta^M_1} \omega^M_z(\{\xi\})d\mu(\xi) = \sum_{n=1}^{n_0} \omega^M_z(\{\xi_n\})\mu(\{\xi_n\}) = \int_{\Delta^M_1} \mu(\{\xi\})d\omega^M_z(\xi)
\]
and
\[
|\mu(\{\xi_n\})| < \infty \quad (n = 1, \ldots, n_0).
\]
Fix a point \(z_0\) of \(R\). We have
\[
\int_{\Delta^M_1} |\mu(\{\xi\})|^q d\omega^M_\infty(\xi) = \sum_{n=1}^{n_0} |\mu(\{\xi_n\})|^q \omega^M_\infty(\{\xi_n\}) < \infty.
\]
Hence \(dim HP(R) \leq n_0\) and by Proposition 3, \(h \in h_q(R)\), that is, \(HP(R) \subset h_q(R)\). Hence \(HP(R) = h_q(R)\). Hence \(dim h_q(R) = dim HP(R) \leq n_0 < \infty\). We get (iii).

Suppose that (iii) holds. Let \(q > 1\). Since \(HP(R)\) and \(h_q(R)\) are linear spaces and \(h_q(R)\) is a subspace of \(HP(R)\), by (iii), we find that \(HP(R) = h_q(R)\). Hence we get (i).

Next we consider \(q = \infty\). Suppose that (i) holds, that is, \(HB(R) = HP(R)\). Let \(q' > 1\). Since \(HB(R) \subset h_{q'}(R) \subset HP(R)\), \(h_{q'}(R) = HP(R)\). By the implication: (i) \(\Rightarrow\) (ii) in the case that \(q \neq \infty\) we get (ii).

Suppose that (ii) holds. Let \(n_0\) be the cardinal number of \(\Delta^M_1\). Set \(\Delta^M_1 = \{\zeta_n\}_{n=1}^{n_0}\). Clearly \(HB(R) \subset HP(R)\). Take any element \(h\) of \(HP(R)\). By Proposition 2 we find a signed measure \(\mu\) on \(\Delta^M\) such that \(\mu(\Delta^M \setminus \Delta^M_1) = 0\) and \(h(z) = \int_{\Delta^M_1} \omega^M_z(\{\xi\})d\mu(\xi)\). By (ii) we have
\[
h(z) = \int_{\Delta^M_1} \omega^M_z(\{\xi\})d\mu(\xi) = \sum_{n=1}^{n_0} \mu(\{\xi_n\})\omega^M_z(\{\xi_n\})
\]
Hence \(dim HP(R) \leq n_0\) and \(h \in HB(R)\), that is, \(HP(R) \subset HB(R)\). Hence \(HP(R) = HB(R)\). Hence \(dim HB(R) = dim HP(R) \leq n_0 < \infty\). We get (iii).
Suppose that (iii) holds. Since $HP(R)$ and $HB(R)$ are linear spaces and $HB(R)$ is a subspace of $HP(R)$, by (iii), we find that $HP(R) = HB(R)$. Hence we get (i).

Therefore we have the desired result.

References

いつ双曲のリーマン面上の異なる指数をもつ調和ハーディ
空間は同一の集合になるか?

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要 旨

双曲的（グリーン関数が存在する）リーマン面とする。1 ≤ p < q ≦ ∞ を仮定する。この論文では、調和ハーディ空間  \( h_p(R) \) と \( h_q(R) \) が同一の集合であるための特徴づけを \( R \) のマルチン境界 \( \Delta^M \) の言葉で与える。

\( \Delta^M \) を \( R \) のミニマルマルチン境界とする。\( p > 1 \) の場合、\( h_p(R) \) と \( h_q(R) \) が同一の集合であるための必要十分条件は \( \Delta^M \) の部分集合 \( N \) が存在して、その \( \Delta^M \) 上の調和測度は 0 で、\( \Delta^M \setminus N \) が有限個の \( \Delta^M \) 上の調和測度が正の点からなることである。\( p = 1 \) である場合、\( h_1(R) \) と \( h_q(R) \) が同一の集合であるための必要十分条件は \( \Delta^M \) が有限個の \( \Delta^M \) 上の調和測度が正の点からなることである。

キーワード：双曲的リーマン面、調和ハーディ空間、マルチン境界、ミニマルマルチン境界、調和測度