A p-adic analytic family of the D-th Shintani lifting for a Coleman family and congruences between the central L-values

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Abstract

We will construct a p-adic analytic family of D-th Shintani lifting generalized by Kojima and Tokuno for a Coleman family. Consequently, we will have a p-adic L-function which interpolates the central L-values attached to a Coleman family and obtain a congruence between the central L-values. Focusing on the case of p-ordinary, we will obtain two applications. One of them states that a congruence between Hecke eigenforms of different weights sufficiently close, p-adically, derives a congruence between their central L-values. The other one is about the Goldfeld conjecture in analytic number theory. We will show that there exists a primitive form satisfying the conjecture for each even weight sufficiently close to 2, 3-adically, thanks to a result of Vatsal.

Keywords: modular form, central L-value, p-adic L-function, Coleman family, Shintani lifting, modular symbol
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1. Introduction

Hida is apparently the first to establish a theory of p-adic interpolation of modular forms of half-integral weight in [10]. He constructed Λ-adic cusp forms of half-integral weight for $\mathrm{SL}(2)/\mathbb{Q}$ and proved a p-adic interpolation of Waldspurger's formula ([32, Corollary 2]) by using the Shimura correspondence. His result is essentially generalized by Ramsey to the case of finite slope in [25]. The results of Ramsey are not constrained to the setting individual families but apply more broadly to the eigencurve. On the other hand, after Hida's work in [10], Stevens established p-adic interpolation of the classical Shintani lifting for a Hida family ([29]). His result is essentially generalized by Park to the case of finite slope in [23]. However, their two results on the classical Shintani lifting leave some room for improvement since the error term of p-adic interpolation is not necessarily a p-adic unit (see Remark 5.8). The significant problem for p-adic interpolation is to deal with the error term of interpolation. To see this, let f be a function whose values at integer points are algebraic integers and F a p-adic analytic function that has the interpolation property for any k in a neighborhood in the domain of F, $F(k) = e_k f(k)$ with some error term $e_k \neq 0$. Assume that the values of f and e_k are contained in the p-adic integer ring $\mathbb{Z}_p$ for each k in some neighborhood B for simplicity. This implies that for $k, k' \in B$, we have $e_k f(k) \equiv e_{k'} f(k') \pmod{p}$. The problem is that the obtained congruence may be trivial if both e_k and $e_{k'}$ are not p-adic units. In [16], Kohnen and Zagier proved an explicit Waldspurger's formula by using the D-th Shintani lifting for a fundamental discriminant D. We remark that the D-th Shintani lifting coincides with the classical Shintani lifting when $D = 1$ at least for the full modular case ([16, Corollary 8]). The main purpose of this paper is to present an improvement of Park's construction of a p-adic family of the classical Shintani lifting for a Coleman family (see Theorem 5.7) and interpolate the central L-values attached to primitive forms lying in a Coleman family (see Corollary 5.9).

Notation and terminology. Throughout the paper, we fix an odd prime p, a positive integer N satisfying $(N, 2p) = 1$ and a non-negative rational number α. We assume that $Np \geq 4$ to ensure that $\Gamma_1(Np)$ is torsion-free. We denote by $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ an algebraic closure of the rational number field $\mathbb{Q}$, and the p-adic number field $\mathbb{Q}_p$, respectively. Let $\mathbb{C}$ be the complex number field and $\mathbb{C}_p$ the p-adic completion of $\bar{\mathbb{Q}}_p$. We fix two embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, and an isomorphism $\mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ which commutes with i_∞ and i_p. Let ord_p be the normalized p-adic additive valuation on $\mathbb{C}_p$ so that $\mathrm{ord}_p(p) = 1$ and $|\cdot|_p$ the absolute value given by ord_p. For $z \in \mathbb{C}$, we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$ and put $z^{k/2} := (\sqrt{z})^k$ for each integer k. We denote by $\Gamma_0(M)$ the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices whose left lower entry is divisible by M. We denote by $S_k(M, \varepsilon)$ the space of $\Gamma_0(M)$-cusp forms of weight k with a Dirichlet character ε modulo M. We denote by $S_k^{\mathrm{new}}(M, \varepsilon)$ the orthogonal complement of the subspace of old forms of level N in $S_k(M, \varepsilon)$ with respect to the Petersson inner product. For a modular form f, we denote by $a_n(f)$ the n-th Fourier coefficient of f and put $L(s, f) := \sum_{n \geq 1} a_n(f) n^{-s}$. We call $f \in S_k(M, \varepsilon)$ a *Hecke eigenform* of level M if f satisfies $f|T_n = a_n(f)f$ for the usual Hecke operators T_n on $S_k(M, \varepsilon)$ for all positive integers n. We refer to a Hecke eigenform of level M in $S_k^{\mathrm{new}}(M, \varepsilon)$ as a *primitive form* of level M. For a Hecke eigenform $f \in S_k(M, \varepsilon)$, the *$T_p$-slope* of f is defined as $\mathrm{ord}_p(a_p(f))$. We denote by $S_k(M, \varepsilon)_\alpha$ the subspace of $S_k(M, \varepsilon)$ spanned by the generalized eigenspaces for eigenvalues λ of T_p with $\mathrm{ord}_p(\lambda) = \alpha$. Let $\mathbb{Z}[\varepsilon]$ be the ring generated by the values of ε over $\mathbb{Z}$. For a $\mathbb{Z}[\varepsilon]$-algebra R and $q := \exp(2\pi\sqrt{-1}z)$, we put

$$S_k(M, \varepsilon; R)_\alpha := (S_k(M, \varepsilon)_\alpha \cap \mathbb{Z}[\varepsilon][[q]]) \otimes_{\mathbb{Z}[\varepsilon]} R, \tag{1}$$

$$S_k^{\mathrm{new}}(M, \varepsilon; R)_\alpha := (S_k^{\mathrm{new}}(M, \varepsilon) \cap S_k(M, \varepsilon; \mathbb{Z}[\varepsilon])_\alpha) \otimes_{\mathbb{Z}[\varepsilon]} R. \tag{2}$$

For a Hecke eigenform f, we denote by $\mathbb{Q}_f$ the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by the eigenvalues of f for the Hecke operators T_n for all positive integers n and refer to it as the *Hecke field* of f. For a Dirichlet character

χ, we denote by χ_0 the primitive character attached to χ, c_χ the conductor of χ, and $G(\chi_0)$ the Gauss sum of χ_0, i.e., $G(\chi_0) := \sum_{a=0}^{c_\chi-1} \chi_0(a)\exp(2\pi\sqrt{-1}a/c_\chi)$. For $f \in S_k(M,\varepsilon)$ and a pritmitive character ψ, we denote by $f \otimes \psi \in S_k(L, \varepsilon\psi)$ the ψ*-twist* of f defined by $a_n(f \otimes \psi) := \psi(n)a_n(f)$ for all $n \geq 1$, where L is the least common multiple of M, c_ψ^2, and $c_\psi c_\varepsilon$ ([20, Lemma 4.3.10.(2)]). For a non-zero integer a, we let χ_a denote the *Kronecker symbol* $\chi_a(b) := \left(\frac{a}{b}\right)$ defined by [20, (3.1.9)]. We call D a *fundamental discriminant* if D is either 1 or the discriminant of a quadratic field. We denote by $\mathbb{1}$ the trivial Dirichlet character. By $d \parallel n$, we mean $d \mid n$ and $(d, n/d) = 1$.

We state the objectives of the paper. Let $f \in S^{\mathrm{new}}_{2k_0+2}(N,\chi^2)_\alpha$ be a primitive form with $2k_0+1 > \alpha \neq (2k_0+1)/2$, $f^* \in S_{2k_0+2}(Np,\chi^2)_\alpha$ the p*-stabilization*, which is a Hecke eigenform of level Np with the same T_q-eigenvalues as f for any q except for $q = p$ (see (115)), D a fundamental discriminant with $(D, Np) = 1$ and $\chi_D\chi(-1)(-1)^{k_0} = -1$, and K the p-adic completion of the number field obtained by adjoining the values of χ and $\chi(-1)^{1/2}|D|^{1/2}G(\chi_0^{-1})$ to the Hecke field $\mathbb{Q}_{f^*}$. Then there exists a *Coleman family* $\{f^*_{2k+2}\}_k$ passing through f^*, which consists of the p-stabilizations f^*_{2k+2} of each primitive form $f_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N,\chi^2;\mathcal{O}_K)_\alpha$ for each $2k$ in

$$W := \{k \in \mathbb{Z} \mid k \equiv 2k_0 \ (\mathrm{mod}\ (p-1)p^m),\ k+1 > \alpha\}, \tag{3}$$

satisfying $f^*_{2k+2} \equiv f^*_{2k_0+2} = f^* \pmod p$ (see Theorem 4.4). We consider the *D-th Shintani lifting* $\theta^{Np}_{k,\chi,D}(f^*_{2k+2})$, which is a cusp form of half-integral weight $k + 3/2$ in the *Kohnen plus space* (see (12) for $\theta^{Np}_{k,\chi,D}$ and (7) for the Kohnen plus space). Let $\Omega(f^*_{2k+2})^- \in \mathbb{C}_p^\times$ be the period attached to f^*_{2k+2} obtained by the fact that the f^*_{2k+2}-part of a group of modular symbols is free of rank one over the ring of integer $\mathcal{O}_K$ of K (see [13, Proposition 3.3]). By the virtue of cohomological interpretation of the D-th Shintani lifting, we can define the algebraic part of the $|D|$-th Shintani lifting

$$\theta^{\mathrm{alg}}_D(f^*_{2k+2}) := (\Omega(f^*_{2k+2})^-)^{-1} p \cdot \theta^{Np}_{k,\chi,D}(f^*_{2k+2}), \tag{4}$$

has the Fourier coefficients in $\mathcal{O}_K$(Theorem 3.3), where we use our hypothesis $Np \geq 4$ to ensure that $\Gamma_0(Np)$ is torsion-free and identify modular symbols with compactly supported cohomology (see Section 3). We will interpolate a family $\{\theta^{\mathrm{alg}}_D(f^*_{2k+2})\}_k$, p-adically. According to Theorem 5.3, we may take the error terms of the p-adic interpolation as p-adic units. Then, we will prove the main theorem that for k sufficiently close to k_0, p-adically, $\theta^{\mathrm{alg}}_D(f^*_{2k+2})$ is congruent to $\theta^{\mathrm{alg}}_D(f^*)$ modulo p-power, up to a p-adic unit (Theorem 5.7). The remarkable property of the D-th Shintani lifting is that $a_{|D|}(\theta^N_{k,\chi,D}(f_{2k+2}))$ equals $L(k+1, f \otimes \chi_D\chi_0^{-1})$, up to an explicit constant (Theorem 2.4). Since f^*_{2k+2} is not a primitive form of level Np, we cannot immediately find a relation between $a_{|D|}(\theta^{Np}_{k,\chi,D}(f^*_{2k+2}))$ and the central L-value attached to f^*_{2k+2}. However, we fortunately see that $a_{|D|}(\theta^{Np}_{k,\chi,D}(f^*_{2k+2}))$ equals $a_{|D|}(\theta^N_{k,\chi,D}(f_{2k+2}))$, up to the product of $2(1-p^{-1})$ and the p-Euler factor (Proposition 2.10). Then we obtain a congruence between the central L-values attached to f^* and f^*_{2k+2} (Corollary 5.9). The final section of the paper gives two applications under the assumption that $\chi = \mathbb{1}$, $\alpha = 0$, and N is square-free. One of them states that a congruence between Hecke eigenforms of different weights sufficiently close, p-adically, derives a congruence between their central L-values, up to a p-adic unit (Theorem 6.1). The other application is for the Goldfeld conjecture in analytic number theory. To state the conjecture, let f be a primitive form of weight $2k+2$ and D a fundamental discriminant. For a positive real number X, we define the number

$$M_f(X) := \sharp\{|D| \leq X \mid L(k+1, f \otimes \chi_D) \neq 0\}. \tag{5}$$

Then the conjecture states that

$$M_f(X) \gg X, \tag{6}$$

i.e., there exists a positive constant c such that for sufficiently large X we have $M_f(X) > cX$. Currently, it seems that the best estimate in general case is due to Ono and Skinner [22], who showed $M_f(X) \gg X/\log X$ (see [22, Corollary 3]). Suppose that $k+1 \geq 6$ is even. Kohnen [15] proved that there exists a Hecke eigenform $f \in S_{2k+2}(\mathrm{SL}_2(\mathbb{Z}))$ satisfying (6) (see [15, Corollary 1]). Moreover, he pointed out that (6) holds for any Hecke eigenform $f \in S_{2k+2}(\mathrm{SL}_2(\mathbb{Z}))$ (see [15, Corollary 2]) assuming a conjecture of Maeda (see [12, Conjecture 1.2]) with respect to each even integer $k+1 \geq 6$. Vatsal showed that a primitive form f attached to a certain elliptic curve over $\mathbb{Q}$ of conductor N with a rational point of order 3 and good ordinary reduction at 3 satisfies (6). Taking $p = 3$ (and hence $N \geq 3$ by the assumption that N is odd with $Np \geq 4$) in Theorem 5.7, we expand this result into the case of higher weights (Theorem 6.4). Our result may be regarded as a generalization of Kohnen's result in [15] to the case of odd square-free level $N \geq 3$.

2. Kojima and Tokuno's D-th Shintani lifting

2.1. Definition and properties

Let k be a non-negative integer, M an odd positive integer and χ a Dirichlet character modulo M. Put $\tilde{\chi} := \chi_\epsilon \chi$ with $\epsilon := \chi(-1)$. We denote the *Kohnen plus space* by

$$S^+_{k+3/2}(4M, \tilde{\chi}) := \left\{ g \in S^{\mathrm{Sh}}_{k+3/2}(4M, \tilde{\chi}) \mid a_n(g) = 0 \text{ if } \chi(-1)(-1)^{k+1}n \equiv 2, 3 \pmod 4 \right\}, \tag{7}$$

where $S^{\mathrm{Sh}}_{k+2/3}(4M, \tilde{\chi})$ is the space of cusp forms of half-integral weight $k+3/2$ with level $4M$ and a character $\tilde{\chi}$ modulo $4M$ in the sense of Shimura [27, p. 447]. Let D be a fundamental discriminant with $\chi(-1)(-1)^{k+1}D > 0$ and $(D, M) = 1$. For $g \in S^+_{k+3/2}(4M, \tilde{\chi})$ and each prime ℓ, the Hecke operator T_{ℓ^2} is defined by

$$a_n(g|T_{\ell^2}) = a_{\ell^2 n}(g) + \chi_{(-1)^{k+1}n}\tilde{\chi}(\ell)\ell^k a_n(g) + \chi(\ell^2)\ell^{2k-1}a_{n/\ell^2}(g) \tag{8}$$

for any positive integer n with $\chi(-1)(-1)^{k+1}n \equiv 0, 1 \pmod 4$. We define the *$D$-th Shimura lifting* $\mathrm{Sh}^M_{k,\chi,D}$ by

$$\mathrm{Sh}^M_{k,\chi,D}(g) := \sum_{n \geq 1} \left(\sum_{d|n} \chi_D \chi(d) d^k a_{n^2|D|/d^2}(g) \right) q^n \tag{9}$$

(see [17, (3-1)]). As Kohnen pointed out in his paper [14, p. 241, l. 4-9], the image of the D-th Shimura lifting $\mathrm{Sh}^M_{k,\chi,D}$ is contained in the space of cusp forms under the assumption that

$$\text{either } k \geq 1,\ M \text{ is square-free, or cubic-free and } \chi = \mathbb{1}. \tag{10}$$

Then the following theorem is a restatement of [17, Theorem 3.1] including the case of $k \geq 0$.

Theorem 2.1. *We have the commutative diagram:*

$$\begin{array}{ccc} S^+_{k+3/2}(4M, \tilde{\chi}) & \xrightarrow{\mathrm{Sh}^M_{k,\chi,D}} & S_{2k+2}(M, \chi^2) \\ \downarrow{\scriptstyle T_{\ell^2}} & & \downarrow{\scriptstyle T_\ell} \\ S^+_{k+3/2}(4M, \tilde{\chi}) & \xrightarrow{\mathrm{Sh}^M_{k,\chi,D}} & S_{2k+2}(M, \chi^2) \end{array} \tag{11}$$

for all primes ℓ. In this sense, the D-th Shimura lifting $\mathrm{Sh}^M_{k,\chi,D}$ is Hecke equivariant.

Now we define the *D-th Shintani lifting* $\theta^M_{k,\chi,D}$ as the adjoint mapping of $\mathrm{Sh}_{k,\chi,D}$ with respect to the Petersson inner product $\langle\ ,\ \rangle$, i.e.,

$$\langle g, \theta^M_{k,\chi,D}(f)\rangle = \langle \mathrm{Sh}^M_{k,\chi,D}(g), f\rangle \tag{12}$$

for every $g \in S_{k+3/2}(4M, \tilde{\chi})$ and $f \in S_{2k+2}(M, \chi^2)$. Then the D-th Shintani lifting $\theta^M_{k,\chi,D}$ is Hecke equivariant, i.e., $\theta^M_{k,\chi,D}(f)|T_{\ell^2} = \theta^M_{k,\chi,D}(f|T_\ell)$ for all primes ℓ. Whenever we use $\theta^M_{k,\chi,D}$, we assume that (10). Let Δ be a non-zero integer with $\Delta \equiv 0, 1 \pmod 4$. We denote by $[a, b, c]$ the binary quadratic form defined by

$$[a, b, c](X, Y) = aX^2 + bXY + cY^2 \tag{13}$$

and call $b^2 - 4ac$ the *discriminant*. We denote by $\mathcal{L}(\Delta)$ the set of all integral binary quadratic forms with discriminant Δ. For each integer M, we set

$$\mathcal{L}_M(\Delta) := \{[a, b, c] \in \mathcal{L}(\Delta) \mid a \equiv 0 \pmod M\}. \tag{14}$$

We let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ act on $[a, b, c] \in \mathcal{L}_M(\Delta)$ by

$$([a, b, c] \circ \gamma)(X, Y) := [a, b, c]((X, Y)\,{}^t\gamma). \tag{15}$$

Letting $\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, we see that the action above is as follows:

$$[a, b, c] \circ \gamma = [ax^2 + bxz + cz^2, 2axy + byz + bxw + 2czw, ay^2 + byw + cw^2] \tag{16}$$

For each $Q = [a, b, c] \in \mathcal{L}_M(\Delta)$, we associate it with the pair $(\omega_Q, \omega_Q^{'})$ of points in $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{i\infty\}$ given by

$$(\omega_Q, \omega_Q^{'}) := \begin{cases} \left((-b - 2\sqrt{\Delta})/2a,\ (-b + 2\sqrt{\Delta})/2a\right) & \text{if } a \neq 0, \\ (-c/b,\ i\infty) & \text{if } a = 0 \text{ and } b > 0, \\ (i\infty,\ -c/b) & \text{if } a = 0 \text{ and } b < 0, \end{cases} \tag{17}$$

and the oriented geodesic path C_Q defined as the image in $\Gamma_0(M)\backslash\mathfrak{H}$ of the semicircle $a|z|^2 + b\mathrm{Re} z + c = 0$ oriented from ω_Q to $\omega_Q^{'}$. We set $\chi_0(Q) := \chi_0(c)$. A simple verification shows that for each $f \in S_{2k+2}(M, \chi^2)$, the integral

$$I_{k,\chi}(f, Q) := \chi_0(Q) \int_{C_Q} f(z) Q(z, 1)^k dz \tag{18}$$

absolutely converges and depends only on the $\Gamma_0(M)$-orbit of Q in $\mathcal{L}_M(\Delta)$. Then by the same computation as in [17], we have the following explicit expressions of the Fourier coefficients of $\theta^M_{k,\chi,D}$.

Theorem 2.2 ([17, Theorem 3.2]). *For any $f \in S_{2k+2}(M, \chi^2)$ and any $n \in \mathbb{Z}_{>0}$ with $\chi(-1)(-1)^{k+1}n \equiv 0, 1 \pmod 4$. Then*

$$a_n(\theta^M_{k,\chi,D}(f)) = c_{k,\chi,D} \sum_{t \mid c_\chi^{-1} M} \mu\chi_D\chi_0^{-1}(t) t^{-k-1} \gamma^M_{k,\chi,D}(f; n, t), \tag{19}$$

where we put

$$c_{k,\chi,D} := (-1)^{[(k+1)/2]} 2^{k+1} \chi_D(c_\chi) \chi(-1)^{1/2} \chi^{-1}(D) c_\chi^k G(\chi_0^{-1}), \tag{20}$$

$$\Delta_{n,t} := t^2 c_\chi^2 |D| n, \tag{21}$$

$$\gamma^M_{k,\chi,D}(f; n, t) := \sum_{Q \in \mathcal{L}_{tc_\chi M}(\Delta_{n,t})/\Gamma_0(M)} \omega_D(Q) I_{k,\chi}(f, Q), \tag{22}$$

and let $[x]$ *be the greatest integer not greater than* x*,* μ *the Möbius function and* ω_D *the generalized genus character as in [14]. Furthermore, if* $f \in S^{\mathrm{new}}_{2k+2}(M, \chi^2)$*, then*

$$a_n(\theta^M_{k,\chi,D}(f)) = c_{k,\chi,D}\gamma^M_{k,\chi,D}(f; n, 1). \tag{23}$$

Remark 2.3. Since the sum (22) equals the Petersson inner product of f and the oldform of level M for $t \neq 1$, (see [17, (3-16)]), we see that

$$\gamma^M_{k,\chi,D}(f; n, t) = 0 \tag{24}$$

for $t \neq 1$ if f is a newform of level M. This is why we obtain the last assertion in the theorem above.

Suppose that $c_\chi \parallel M$. Let ℓ be a prime factor of M/c_χ, We put $v_\ell := \mathrm{ord}_\ell(M/c_\chi) = \mathrm{ord}_\ell(M)$. Let γ_ℓ be an element in $\mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma_\ell \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \pmod{\ell^{2v_\ell}}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \pmod{(M/\ell^{v_\ell})^2}. \end{cases} \tag{25}$$

We put $\eta_\ell := \gamma_\ell \cdot \mathrm{diag}(\ell^{v_\ell}, 1)$ (see [20, (4.6.21)]). We define the eigenvalue of f for the Atkin-Lehner involution η_ℓ by

$$w_\ell(f) := \chi^2(\ell^{v_\ell}) a_1(f|_{2k+2}\eta_\ell). \tag{26}$$

If $v_\ell = 1$, then we have $a_1(f|_{2k+2}\eta_\ell) = -\chi^{-2}(\ell)\ell^{-k}a_\ell(f)$ by [20, Corollary 4,6,18.(2)] and hence

$$w_\ell(f) = -\ell^{-k}a_\ell(f) \in \{\pm 1\} \tag{27}$$

by [20, Theorem 4.6.17.(2)].

Theorem 2.4 ([17, Theorem 4.2 and (4-12)]). *Let* $f \in S^{\mathrm{new}}_{2k+2}(M, \chi^2)$ *be a primitive form. Suppose that* $c_\chi \parallel M$*. We put*

$$R_D(f) := \prod_\ell \left(1 + \chi_D\chi(\ell^{v_\ell})w_\ell(f)\left(\frac{1 - \chi_D\chi^{-1}(\ell)\ell^{-k-1}a_\ell(f)}{1 - \chi_D\chi(\ell)\ell^{-k-1}a_\ell(f)^c}\right)\right), \tag{28}$$

where $\prod_\ell$ *is taken over all prime factors* ℓ *of* M/c_χ *and* $a_\ell(f)^c$ *is the complex conjugate of* $a_\ell(f)$*. Then*

$$a_{|D|}(\theta^M_{k,D,\chi}(f)) = R_D(f)|D|^{k+1/2}c_\chi^{2k+1}\pi^{-(k+1)}k!L\left(k+1, f \otimes \chi_D\chi_0^{-1}\right), \tag{29}$$

Remark 2.5. Let the notation and the assumption be the same as the theorem above.

1. If $R_D(f) \neq 0$, then $\mathrm{ord}_p(R_D(f)) = 1$.
2. If $\chi^2 = \mathbb{1}$, then the Hecke field of f is totally real by [26, Proposition 1.3], and hence

$$R_D(f) = \prod_\ell \left(1 + \chi_D\chi(\ell^{v_\ell})w_\ell(f)\right). \tag{30}$$

3. If $\chi^2 = \mathbb{1}$ and M/c_χ is square-free, then $R_D(f) \in \{0, 2^{\nu(M/c_\chi)}\}$ by (27), where $\nu(M/c_\chi)$ is the number of distinct prime factors of M/c_χ. In particular, if $\chi = \mathbb{1}$, then the followings are equivalent:
   (a) $R_D(f) \neq 0$.
   (b) $R_D(f) = 2^{\nu(M)}$.
   (c) $\chi_D(\ell) = w_\ell(f)$ for all prime divisors ℓ of M.
   In this case, the formula (29) is nothing but the result of Kohnen in [14] and the sign of the functional equation of $L(s, f \otimes \chi_D)$ is $(-1)^{k+1}\chi_D(-1)$, i.e., if $(-1)^{k+1}\chi_D(-1) = -1$, then $L(k+1, f \otimes \chi_D) = 0$.

2.2. Integral binary quadratic forms on which $\Gamma_0(M)$ acts

We need to prepare more notations for sets of quadratic forms in order to state a key lemma below (Lemma 2.8), which plays an important role in the proof of Proposition 2.10. We refer to [9] for a theory of quadratic forms that we need. We fix a positive integer M and a non-zero integer $\Delta \equiv 1, 0 \pmod 4$ in this subsection. We denoted the set of $\Gamma_0(M)$-primitive quadratic forms of discriminant Δ by

$$\mathcal{L}_M^0(\Delta) := \{[Ma, b, c] \in \mathcal{L}_M(\Delta) \mid (a, b, c) = 1\}. \tag{31}$$

We set

$$S_M(\Delta) := \{\bar{\varrho} \in \mathbb{Z}/2M\mathbb{Z} \mid \varrho^2 \equiv \Delta \pmod{4M}\}. \tag{32}$$

For $\bar{\varrho} \in S_M(\Delta)$, we set

$$\mathcal{L}_{M,\varrho}^0(\Delta) := \{[Ma, b, c] \in \mathcal{L}_M^0(\Delta) \mid b \equiv \varrho \pmod{2M}\}. \tag{33}$$

Note that the $\Gamma_0(M)$-action $\circ$ defined by (16) preserves $\mathcal{L}_{M,\varrho}^0(\Delta)$ and that we have the following decomposition into the disjoint union of $\Gamma_0(M)$-invariant sets:

$$\mathcal{L}_M^0(\Delta) = \bigsqcup_{\bar{\varrho} \in S_M(\Delta)} \mathcal{L}_{M,\varrho}^0(\Delta). \tag{34}$$

We then have the following decomposition into the union of $\Gamma_0(M)$-invariant sets:

$$\mathcal{L}_M(\Delta) = \bigsqcup_{l^2 \mid \Delta} l \cdot \mathcal{L}_M^0(\Delta/l^2) = \bigsqcup_{l^2 \mid \Delta} \bigcup_{\bar{\varrho} \in S_M(\Delta/l^2)} l \cdot \mathcal{L}_{M,\varrho}^0(\Delta/l^2), \tag{35}$$

where the disjoint union $\bigsqcup_{l^2 \mid \Delta}$ is taken over all positive integers l such that $l^2 \mid \Delta$. For parameters M, Δ, ϱ of $\mathcal{L}_{M,\varrho}^0(\Delta)$, we define the greatest common divisor

$$m_\varrho^M := m := \left(M, \varrho, (\varrho^2 - \Delta)/4M\right). \tag{36}$$

Note that the definition (36) depends only on ϱ modulo $2M$. For $[Ma, b, c] \in \mathcal{L}_{M,\varrho}^0(\Delta)$, we have $(M, b, ac) = m$ and $(a, b, c) = 1$, so the two numbers

$$(M, b, a) = m_1 \text{ and } (M, b, c) = m_2 \tag{37}$$

are coprime and $m_1 m_2 = m$. We denote by $\mathcal{L}_{M,\varrho,m_1,m_2}^0(\Delta)$ the set of forms $[Ma, b, c] \in \mathcal{L}_{M,\varrho}^0(\Delta)$ satisfying (37). We then have the following decomposition into the disjoint union of $\Gamma_0(M)$-invariant sets:

$$\mathcal{L}_{M,\varrho}^0(\Delta) = \bigsqcup_{m_1, m_2} \mathcal{L}_{M,\varrho,m_1,m_2}^0(\Delta), \tag{38}$$

where $\bigsqcup_{m_1,m_2}$ is taken over all pairs (m_1, m_2) of positive integers m_1, m_2 satisfying $(m_1, m_2) = 1$ and $m = m_1 m_2$. Summarizing, we have the following decomposition of $\mathcal{L}_M(\Delta)$ into the union of $\Gamma_0(M)$-invariant sets:

$$\mathcal{L}_M(\Delta) = \bigsqcup_{l^2 \mid \Delta} \bigcup_{\bar{\varrho} \in S_M(\Delta/l^2)} \bigsqcup_{m_1, m_2} l \cdot \mathcal{L}_{M,\varrho,m_1,m_2}^0(\Delta/l^2), \tag{39}$$

where $\bigsqcup_{m_1,m_2}$ is taken over all pairs (m_1, m_2) of positive integers m_1, m_2 satisfying $(m_1, m_2) = 1$ and

$$(M, \varrho, (\varrho^2 - \Delta/l^2)/4M) = m_1 m_2. \tag{40}$$

We put $\mathcal{L}^0(\Delta) := \mathcal{L}_1^0(\Delta)$.

Proposition 2.6 ([9, Proposition, p.505]). *Let M_1 and M_2 be positive integers satisfying $M = M_1M_2$ and $(M_1, M_2) = (m_1, M_2) = (m_2, M_1) = 1$. Then, the mapping $[Ma, b, c] \mapsto [M_1a, b, M_2c]$ induces a bijection*

$$\mathcal{L}^0_{M,\varrho,m_1,m_2}(\Delta)/\Gamma_0(M) \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{L}^0(\Delta)/\operatorname{SL}_2(\mathbb{Z}). \tag{41}$$

We prove the following lemma needed in the proof of Proposition 2.10.

Lemma 2.7. *Let $\varrho \in S_{Np}(\Delta)$ and $\varrho' \in S_N(\Delta)$ and let m, m' be positive integers with $m \parallel m^{Np}_{\varrho}$ and $m' \parallel m^{N}_{\varrho'}$. The map $[a, b, c] \mapsto [a, b, c]$ induces a bijection*

$$\mathcal{L}^0_{Np,\varrho,m,1}(\Delta)/\Gamma_0(Np) \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{L}^0_{N,\varrho',m',1}(\Delta)/\Gamma_0(N). \tag{42}$$

Moreover, if $(m.p) = 1$, then $\tau : [a, b, c] \mapsto [a/p, b, pc]$ induces a bijection between the same spaces as above.

Proof. Taking $(Np, 1)$ and (N, p) as the ordered pairs (M_1, M_2) in Proposition 2.6 for $M := Np$, we see that both mappings induce two bijections

$$\mathcal{L}^0_{Np,\varrho,m,1}(\Delta)/\Gamma_0(Np) \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{L}^0(\Delta)/\operatorname{SL}_2(\mathbb{Z}) \tag{43}$$

by Proposition 2.6. On the other hand, taking $(N, 1)$ as the ordered pair (M_1, M_2) in Proposition 2.6 for $M := N$, we see that the mapping $[a, b, c] \mapsto [a, b, c]$ induces a bijection

$$\mathcal{L}^0(\Delta)/\operatorname{SL}_2(\mathbb{Z}) \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{L}^0_{N,\varrho',m',1}(\Delta)/\Gamma_0(N) \tag{44}$$

by Proposition 2.6. Composing these maps, we obtain the assertion.

Assume that Δ is a perfect square and let δ be a positive integer such that $\Delta = \delta^2$. For a positive integer M' with $M' \parallel M$, we define a map $w_{M'} : S_M(\Delta) \to S_M(\Delta)$ by

$$w_{M'}(\varrho) \equiv \begin{cases} \varrho & (\operatorname{mod} 2M/M'), \\ -\varrho & (\operatorname{mod} M'). \end{cases} \tag{45}$$

Similarly to Atkin-Lehner involutions $W_{M'}$ on quadratic forms in [9, Section 1], these maps $w_{M'}$ are bijections and satisfy the relation $w_{M'} \circ w_{M''} = w_{M'M''/(M',M'')^2}$, so they form a group of order 2^t, where t is the number of distinct prime factors of M.

Lemma 2.8. *Let c be a positive integer with $c \parallel M$ and d an integer with $(d, M) = 1$. Then we have the decomposition into the disjoint union of $\Gamma_0(M)$-invariant sets*

$$\mathcal{L}_{cM}(c^2d^2) = \bigsqcup_{l \mid cd} \bigsqcup_{M' \parallel c^{-1}M} l \cdot \mathcal{L}^0_{M,w_{M'}(cd/l),c/(c,l),1}(c^2d^2/l^2), \tag{46}$$

where $\bigsqcup_{l \mid cd}$ and $\bigsqcup_{M' \parallel c^{-1}M}$ is taken over all positive divisors l of cd and all positive integers M' with $M' \parallel c^{-1}M$, respectively.

Proof. We put $\delta := cd$ and $\Delta := \delta^2$ for short. For a positive divisor l of δ and $\varrho \in S_M(\Delta/l^2)$, we denote by $m(l, \varrho)$ the greatest common divisor of M, ϱ, and $(\varrho^2 - \Delta/l^2)/4M$. From

$$\mathcal{L}_M(\Delta) = \bigsqcup_{l \mid \delta} \bigcup_{\varrho \in S_M(\Delta/l^2)} \bigsqcup_{m \parallel m(l,\varrho)} l \cdot \mathcal{L}^0_{M,\varrho,m,m(l,\varrho)/m}(\Delta/l^2)$$

((39)), we see that

$$\mathcal{L}_{cM}(\Delta) = \bigsqcup_{l|\delta} \bigcup_{\varrho \in S_M(\Delta/l^2)} \mathcal{L}_{cM}(\Delta)_{l,\varrho}, \text{ where } \mathcal{L}_{cM}(\Delta)_{l,\varrho} := \bigsqcup_{\substack{m \| m(l,\varrho) \\ lm \equiv 0 \pmod{c}}} l \cdot \mathcal{L}^0_{M,\varrho,m,m(l,\varrho)/m}(\Delta/l^2).$$

Since $lm \equiv 0 \pmod{c}$ implies $m(l,\varrho) \equiv 0 \pmod{c/(c,l)}$ for $m \parallel m(l,\varrho)$, we see that the union runs over $\varrho \in S_M(\Delta/l^2)$ such that $m(l,\varrho) \equiv 0 \pmod{c/(c,l)}$ we have Via the natural bijection from $G := \{M' \in \mathbb{Z}_{>0} \mid M' \parallel M\}$ into the group of $w_{M'}$'s, we may regard G as a group and G acts on the set $S_M(\Delta/l^2)$ for any positive divisor l of δ. For a prime divisor q of M, we put $v_q := \mathrm{ord}_q(M)$, $n := [v_q/2]$, and,

$$R_q := \{mp^{n'} \mid m \in \mathbb{Z}, 0 \le m \le (q^n - 1)/2\} \text{ with } n' := \begin{cases} n & \text{if } v_q \text{ is even,} \\ n+1 & \text{if } v_q \text{ is odd.} \end{cases} \tag{47}$$

Notice that $R_q \cup (-R_q)$ is a complete system of representatives for $\{\bar{x} \in \mathbb{Z}/q^{v_q}\mathbb{Z} \mid x^2 \equiv 0 \pmod{q^{v_q}}\}$. Let S be the set of prime divisors q of M such that $\Delta/l^2 \equiv 0 \pmod{q^{v_q}}$. For $r = (r_q)_q \in \Pi_{q \in S} R_q$, we let ϱ_r be an element in $S_M(\Delta/l^2)$ such that for any prime factor q of $2M$,

$$\varrho_r \equiv \begin{cases} r_q & \pmod{q^{v_q}} \text{ if } q \in S, \\ \delta/l & \pmod{q^{v_q}} \text{ if } q \notin S. \end{cases} \tag{48}$$

We then have the G-orbit decomposition $S_M(\Delta/l^2) = \bigsqcup_{(r_q)_q \in \Pi_{q \in S} R_q} G \cdot \varrho_r$. Note that $m(l,\varrho) = m(l,\varrho_r)$ if $\varrho \in G \cdot \varrho_r$ and that for any $\varrho \in S_M(\Delta/l^2)$, we see that $m(l,\varrho) \equiv 0 \pmod{c/(c,l)}$ if and only if $\varrho \in G \cdot \delta/l$, and in this case $m(l,\varrho) = c/(c,l)$. We thus have

$$\bigcup_{\substack{\varrho \in S_M(\Delta/l^2) \\ m(l,\varrho) \equiv 0 \pmod{c/(c,l)}}} \mathcal{L}_{cM}(\Delta)_{l,\varrho} = \bigcup_{\varrho \in G \cdot \delta/l} \mathcal{L}_{cM}(\Delta)_{l,\varrho} = \bigcup_{\varrho \in G \cdot \delta/l} l \cdot \mathcal{L}^0_{M,\varrho,c/(c,l),1}(\Delta/l^2).$$

Here, for $\varrho_1, \varrho_2 \in G \cdot \delta/l$, we see that the intersection of $l \cdot \mathcal{L}^0_{M,\varrho_1,c/(c,l),1}(\Delta/l^2)$ and $l \cdot \mathcal{L}^0_{M,\varrho_2,c/(c,l),1}(\Delta/l^2)$ is non-empty if and only if $\varrho_1 \equiv \varrho_2 \pmod{2M/c}$. Therefore, we have

$$\bigcup_{\varrho \in G \cdot \delta/l} l \cdot \mathcal{L}^0_{M,\varrho,c/(c,l),1}(\Delta/l^2) = \bigsqcup_{M' \| c^{-1}M} l \cdot \mathcal{L}^0_{M,w_{M'}(\delta/l),c/(c,l),1}(\Delta/l^2).$$

2.3. Relationship between $a_{|D|}(\theta^{Np}_{k,\chi,D}(f^*))$ and $a_{|D|}(\theta^{N}_{k,\chi,D}(f))$

Lemma 2.9. *For any $f^* \in S_{2k+2}(Np,\chi^2)$ and any $n \in \mathbb{Z}_{>0}$ with $\chi(-1)(-1)^{k+1}n \equiv 0, 1 \pmod{4}$, we have*

$$a_n(\theta^{Np}_{k,\chi,D}(f^*)) = \left(1 - p^{-1}\right) c_{k,\chi,D} \sum_{t | c_\chi^{-1} N} \mu\chi_D\chi_0^{-1}(t) t^{-k-1} \gamma^{Np}_{k,\chi,D}(f^*; n, t), \tag{49}$$

where recall that $c_{k,\chi,D}$, $\Delta_{n,t}$, and $\gamma^{Np}_{k,\chi,D}(f; n,t)$ are given by (20), (21), and (22), respectively.

Proof. We put $a(t) := \mu\chi_D\chi^{-1}(t)t^{-k-1}\gamma^{Np}_{k,\chi,D}(f^*; n,t)$ for short. We see that

$$\sum_{t | c_\chi^{-1} Np} a(t) = \sum_{t | c_\chi^{-1} N} \left(a(t) + a(pt)\right). \tag{50}$$

By Theorem 2.2, it suffices to prove $a(pt) = -p^{-1}a(t)$. Let $t \mid c_\chi^{-1}N$ and $Q \in \mathcal{L}_{ptc_\chi Np}(\Delta_{n,pt})/\Gamma_0(Np)$. Notice that the coefficients of the quadratic form Q are divisible by p. Since $\omega_D(Q) = \chi_D(p)\omega_D(p^{-1}Q)$ and $I_{k,\chi}(f^*, Q) = \chi(p)p^k I_{k,\chi}(f^*, p^{-1}Q)$, we see that

$$\gamma_{k,\chi,D}^{Np}(f^*; n, pt) = \chi_D\chi(p)p^k\gamma_{k,\chi,D}^{Np}(f^*; n, t).$$

We thus have $a(pt) = \mu\chi_D\chi^{-1}(pt)(pt)^{-k-1} \cdot \chi_D\chi(p)p^k\gamma_{k,\chi,D}^{Np}(f^*; n, t) = -p^{-1}a(t)$.

For a formal power series $\sum_{n\geq 0} a(n)q^n$, we define

$$\left(\sum_{n\geq 0} a(n)q^n\right)|V_p := \sum_{n\geq 0} a(n)q^{pn}. \tag{51}$$

Proposition 2.10. *Let $f \in S_{2k+2}^{new}(N, \chi^2)$ be a primitive form with $c_\chi \parallel N$ and D a fundamental discriminant with $\chi(-1)(-1)^{k+1}D > 0$ and $(D, Np) = 1$. We put $f^* := f - \beta \cdot f|V_p \in S_{2k+2}(Np, \chi^2)$ with $\beta \in \mathbb{C}$. Then,*

$$a_{|D|}(\theta_{k,\chi,D}^{Np}(f^*)) = 2\left(1 - p^{-1}\right)\left(1 - \chi_D\chi^{-1}(p)p^{-k-1}\beta\right) \cdot a_{|D|}(\theta_{k,\chi,D}^{N}(f)). \tag{52}$$

Proof. By Lemma 2.9 and Theorem 2.2, we have

$$a_{|D|}(\theta_{k,\chi,D}^{Np}(f^*)) = (1 - p^{-1})c_{k,\chi,D} \sum_{t|c_\chi^{-1}N} \mu\chi_D\chi^{-1}(t)t^{-k-1}\gamma_{k,\chi,D}^{Np}(f^*; |D|, t), \tag{53}$$

$$\begin{aligned} a_{|D|}(\theta_{k,\chi,D}^{N}(f)) &= c_{k,\chi,D} \sum_{t|c_\chi^{-1}N} \mu\chi_D\chi^{-1}(t)t^{-k-1}\gamma_{k,\chi,D}^{N}(f; |D|, t) \\ &= c_{k,\chi,D} \cdot \gamma_{k,\chi,D}^{N}(f; |D|, 1), \end{aligned} \tag{54}$$

where the last equation is due to (24). We put $I_Q(f) := \omega_D(Q)I_{k,\chi}(f, Q)$ for short. Remember that, from the notation (22), we have

$$\gamma_{k,\chi,D}^{Np}(f^*; |D|, t) = \sum_{Q\in\mathcal{L}_{tc_\chi Np}(\Delta_{|D|,t})/\Gamma_0(Np)} I_Q(f^*), \tag{55}$$

$$\gamma_{k,\chi,D}^{N}(f; |D|, t) = \sum_{Q\in\mathcal{L}_{tc_\chi N}(\Delta_{|D|,t})/\Gamma_0(N)} I_Q(f), \tag{56}$$

Note that

$$\gamma_{k,\chi,D}^{Np}(f^*; |D|, t) = \gamma_{k,\chi,D}^{Np}(f; |D|, t) - \beta \cdot \gamma_{k,\chi,D}^{Np}(f|V_p; |D|, t). \tag{57}$$

We put

$$a := \sum_{t|c_\chi^{-1}N} \mu\chi_D\chi^{-1}(t)t^{-k-1}\gamma_{k,\chi,D}^{Np}(f^*; |D|, t). \tag{58}$$

Then $a_{|D|}(\theta_{k,\chi,D}^{Np}(f^*)) = (1-p^{-1})c_{k,\chi,D} \cdot a$. Let t be a positive and square-free divisor of N/c_χ so that $tc_\chi \parallel N$. We put $\delta_t := tc_\chi D$ for short so that $\Delta_{|D|,t} = \delta_t^2$. Taking (Np, tc_χ, D) as the ordered triple (M, c, d) in Lemma 2.8, we have

$$\mathcal{L}_{tc_\chi Np}(\Delta_{|D|,t}) = \bigsqcup_{l|\delta_t} \bigsqcup_{M'\|(tc_\chi)^{-1}Np} \mathcal{L}(l, M') = \mathcal{L}^{(p)} \sqcup \mathcal{L}_p, \tag{59}$$

where we put

$$\mathcal{L}(l, M') := l \cdot \mathcal{L}^0_{Np, w_{M'}(\delta_t/l), tc_\chi/(tc_\chi, l), 1}(\Delta_{|D|,t}/l^2) \tag{60}$$

$$\mathcal{L}^{(p)} := \bigsqcup_{l|\delta_t} \bigsqcup_{M' \| (tc_\chi)^{-1}N} \mathcal{L}(l, M') \text{ and } \mathcal{L}_p := \bigsqcup_{l|\delta_t} \bigsqcup_{M' \| (tc_\chi)^{-1}N} \mathcal{L}(l, pM'). \tag{61}$$

We thus have

$$\gamma^{Np}_{k,\chi,D}(f; |D|, t) = \sum_{Q \in \mathcal{L}^{(p)}/\Gamma_0(Np)} I_Q(f) + \sum_{Q \in \mathcal{L}_p/\Gamma_0(Np)} I_Q(f), \tag{62}$$

$$\gamma^{Np}_{k,\chi,D}(f|V_p; |D|, t) = \sum_{Q \in \mathcal{L}^{(p)}/\Gamma_0(Np)} I_Q(f|V_p) + \sum_{Q \in \mathcal{L}_p/\Gamma_0(Np)} I_Q(f|V_p). \tag{63}$$

Taking (N, tc_χ, D) as the ordered triple (M, c, d) in Lemma 2.8, we have

$$\mathcal{L}_{tc_\chi N}(\Delta_{|D|,t}) = \bigsqcup_{l|\delta_t} \bigsqcup_{M' \| (tc_\chi)^{-1}N} l \cdot \mathcal{L}^0_{N, w_{M'}(\delta_t/l), tc_\chi/(tc_\chi, l), 1}(\Delta_{|D|,t}/l^2). \tag{64}$$

By Lemma 2.7, both mappings $[a, b, c] \mapsto [a, b, c]$ and $\tau : [a, b, c] \mapsto [a/p, b, pc]$ induce two bijections

$$\mathcal{L}^{(p)}/\Gamma_0(Np) \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{L}_{tc_\chi N}(\Delta_{|D|,t})/\Gamma_0(N) \text{ and } \mathcal{L}_p/\Gamma_0(Np) \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{L}_{tc_\chi N}(\Delta_{|D|,t})/\Gamma_0(N). \tag{65}$$

Via two bijections (65) induced by $[a, b, c] \mapsto [a, b, c]$, we have

$$\sum_{Q \in \mathcal{L}^{(p)}/\Gamma_0(Np)} I_Q(f) = \sum_{Q \in \mathcal{L}_p/\Gamma_0(Np)} I_Q(f) = \gamma^N_{k,\chi,D}(f; |D|, t) \tag{66}$$

and by (62), we have

$$\gamma^{Np}_{k,\chi,D}(f; |D|, t) = 2 \cdot \gamma^N_{k,\chi,D}(f; |D|, t). \tag{67}$$

Via two bijections (65) induced by $\tau : [a, b, c] \mapsto [a/p, b, pc]$, we see that both $\sum_{Q \in \mathcal{L}^{(p)}/\Gamma_0(Np)} I_Q(f|V_p)$ and $\sum_{Q \in \mathcal{L}_p/\Gamma_0(Np)} I_Q(f|V_p)$ coincide with

$$\sum_{Q \in \mathcal{L}_{tc_\chi N}(\Delta_t)/\Gamma_0(N)} I_{\tau^{-1}(Q)}(f|V_p). \tag{68}$$

Here, by [9, Proposition 1 (Multiplicativity) and (Explicit formula)], we have $\omega_D(\tau^{-1}(Q)) = \chi_D(p)\omega_D(Q)$ and by a simple calculation, we have

$$\chi_0(\tau^{-1}(Q)) = \chi^{-1}(p)\chi_0(Q), \tag{69}$$

$$\int_{C_{\tau^{-1}(Q)}} f(pz)\tau^{-1}(Q)(z, 1)^k dz = p^{-k-1} \int_{C_Q} f(z)Q(z, 1)^k dz. \tag{70}$$

Indeed, we see that the last equation as follows: Put $[a, b, c] := Q$. Then

$$\begin{aligned}
\int_{C_{\tau^{-1}(Q)}} f(pz)\tau^{-1}(Q)(z, 1)^k dz &= \int_{\omega_{\tau^{-1}(Q)}}^{\omega'_{\tau^{-1}(Q)}} f(pz)(paz^2 + bz + c/p)^k dz \\
&= p^{-k} \int_{p^{-1}\omega_Q}^{p^{-1}\omega'_Q} f(pz)(a(pz)^2 + b(pz) + c)^k dz \\
&= p^{-k} \int_{\omega_Q}^{\omega'_Q} f(z)(az^2 + bz + c)^k p^{-1} dz = p^{-k-1} \int_{C_Q} f(z)Q(z, 1)^k dz,
\end{aligned}$$

where at the second equation from the bottom, we have made use of the transformation law with respect to $z \mapsto p^{-1}z$. We thus have $I_{\tau^{-1}(Q)}(f|V_p) = \chi_D\chi^{-1}(p)p^{-k-1}I_Q(f)$, and hence (68) coincides with

$$\chi_D\chi^{-1}(p)p^{-k-1}\gamma^N_{k,\chi,D}(f;|D|,t). \tag{71}$$

By (63), we have

$$\gamma^{Np}_{k,\chi,D}(f|V_p;|D|,t) = 2\cdot\chi_D\chi^{-1}(p)p^{-k-1}\gamma^N_{k,\chi,D}(f;|D|,t). \tag{72}$$

From (57), (67) and (72), we have

$$\begin{aligned} a &= \sum_{t|c_\chi^{-1}N} \mu\chi_D\chi^{-1}(t)t^{-k-1}2\left(1-\chi_D\chi^{-1}(p)p^{-k-1}\beta\right)\gamma^N_{k,\chi,D}(f;|D|,t) \\ &= 2\left(1-\chi_D\chi^{-1}(p)p^{-k-1}\beta\right)\gamma^N_{k,\chi,D}(f;|D|,1), \end{aligned} \tag{73}$$

where the last equation is due to (24).

3. Cohomological interpretation of the D-th Shintani lifting

In this section, we will construct the cohomological D-th Shintani lifting $\Theta^{Np}_{k,\chi,D}$ satisfying the following commutative diagram:

$$\begin{array}{ccccc} H^1_c(\Gamma_0(Np), L(2k,\chi^2;\mathbb{C}))^- & \xrightarrow[\text{Ash-Stevens}]{\sim} & \mathrm{Symb}_{\Gamma_0(Np)}(L(2k,\chi^2;\mathbb{C}))^- & \xrightarrow{\Theta^{Np}_{k,\chi,D}} & \mathbb{C}[[q]] \\ \uparrow{\scriptstyle\text{Manin-Drinfeld}} & & & & \uparrow{\scriptstyle q\text{-expansion}} \\ H^1_p(\Gamma_0(Np), L(2k,\chi^2;\mathbb{C}))^- & \xleftarrow[\text{Eichler-Shimura}]{\sim} & S_{2k+2}(Np,\chi^2) & \xrightarrow{\theta^{Np}_{k,\chi,D}} & S^+_{k+3/2}(4Np,\tilde{\chi}), \end{array}$$

where all arrows are Hecke equivariant $\mathbb{C}$-homomorphisms and we concentrate on the minus parts because of $\Theta^{Np}_{k,\chi,D}(\mathrm{Symb}_{\Gamma_0(Np)}(L(2k,\chi^2;\mathbb{C}_p))^+) = 0$.

3.1. Modular symbols and the Eichler-Shimura isomorphism

Let Δ_0 be a subsemigroup of $\mathrm{M}_2(\mathbb{Z})\cap\mathrm{GL}_2(\mathbb{Q})$ containing $\Gamma_0(M)$. Let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of divisors of degree 0 supported on the rational cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}\cup\{i\infty\}$ of the complex upper half plane $\mathfrak{H}$. We let Δ_0 act on $\mathfrak{H}$ by fractional linear transformations, i.e.,

$$\gamma z := \begin{cases} (az+b)(cz+d)^{-1} \text{ if } \det(\gamma) > 0, \\ (a\bar{z}+b)(c\bar{z}+d)^{-1} \text{ if } \det(\gamma) < 0, \end{cases} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \in \mathfrak{H}\right). \tag{74}$$

This induces a natural action of Δ_0 on $\mathfrak{H}^* := \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Q})$. Then Δ_0 acts on $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ by linear fractional transformations. Let R be a commutative ring and E a left $R[\Delta_0]$-module. We let $\gamma\in\Delta_0$ acts on $\Phi\in\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), E)$ by

$$(\Phi|\gamma)(D) := \gamma\Phi(\gamma D). \tag{75}$$

Then the abstract Hecke algebra $R[\Gamma_0(M)\backslash\Delta_0/\Gamma_0(M)]$ with respect to the Hecke pair $(\Gamma_0(M),\Delta_0)$ acts on the group of E-valued *modular symbols* over $\Gamma_0(M)$:

$$\mathrm{Symb}_{\Gamma_0(M)}(E) := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), E)^{\Gamma_0(M)}. \tag{76}$$

Let $\tilde{E}$ be the locally constant sheaf on the open modular curve $Y := \Gamma_0(M)\backslash\mathfrak{H}$ attached to E. Assume that

$$\text{the orders of the torsion elements of } \Gamma_0(M) \text{ act invertibly on } E. \tag{77}$$

Then by [3, Proposition 4.2], there exists a Hecke equivariant canonical isomorphism

$$H^1_c(Y, \tilde{E}) \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(M)}(E). \tag{78}$$

Throughout the paper, we will identify the group of compactly supported cohomology classes with the group of modular symbols under the assumption that (77). Note that (77) holds if either E is a vector space over a field of characteristic 0, E is a $\mathbb{Z}_p$-module with $p \geq 5$, or $\Gamma_0(M)$ is torsion-free. Fix a point $x_0 \in \mathbb{P}^1(\mathbb{Q})$. The natural map $\mathrm{Symb}_{\Gamma_0(M)}(E) \to H^1(\Gamma_0(M), E)$ sends a modular symbol Φ to the cohomology class represented by the 1-cocycle $\gamma \mapsto \Phi(\{\gamma x_0\} - \{x_0\})$. This map yields a Hecke equivariant epimorphism

$$\mathrm{Symb}_{\Gamma_0(M)}(E) \twoheadrightarrow H^1_p(\Gamma_0(M), E). \tag{79}$$

The matrix $\iota := \mathrm{diag}(1, -1)$ induces natural involutions on one of the above cohomology groups H, and each of cohomology groups H is decomposed into $\pm$-eigenmodules $H = H^+ \oplus H^-$ if 2 acts invertibly on the coefficient module of H. Indeed, each cohomology class Φ decomposes as $\Phi = \Phi^+ + \Phi^-$, where $\Phi^\pm := 2^{-1}(\Phi \pm \Phi|\iota)$. For a non-negative integer n, let $L(n, R)$ be the R-module of homogeneous polynomials in (X, Y) of degree n with coefficients in R. Let ε be an R-valued Dirichlet character modulo M. We denote by $L(n, \varepsilon; R)$ the $R[\Gamma_0(M)]$-module $L(n, R)$ endowed with the ε-twisted action, i.e., for $\gamma \in \Gamma_0(M)$ and $P(X, Y) \in L(n, \varepsilon; R)$,

$$(\gamma P)(X, Y) = \varepsilon(\gamma) P((X, Y)^t\gamma), \tag{80}$$

where $\varepsilon(\gamma)$ is the value of ε at the lower right entry of γ. Suppose that $n!$ is invertible in R. We define a pairing $[\ ,\ ] : L(n, R) \times L(n, R) \to R$ by

$$\left[\sum_{i=0}^{n} a_j X^{n-i} Y^j, \sum_{i=0}^{n} b_i X^{n-i} Y^i\right] := \sum_{i=0}^{n} (-1)^i \binom{n}{i}^{-1} a_i b_{n-i}. \tag{81}$$

We use the following two properties later:

$$[(aX - bY)^n, P(X, Y)] = (-1)^n P(b, a) \tag{82}$$

$$[\gamma P, \gamma Q] = \det \gamma^n [P, Q] \tag{83}$$

for $a, b \in R$, $P, Q \in L(n, R)$ and $\gamma \in \mathrm{M}_2(R)$. If K is a field of characteristic zero, then by the Manin-Drinfeld principle there exists a unique Hecke equivariant section

$$s_{k,\varepsilon} : H^1_p(\Gamma_0(M), L(k, \varepsilon; K)) \hookrightarrow \mathrm{Symb}_{\Gamma_0(M)}(L(k, \varepsilon; K)) \tag{84}$$

of the surjection (79). For each cusp form $f \in S_{k+2}(M, \varepsilon)$, we define the $L(k, \varepsilon; \mathbb{C})$-valued differential form on $\mathfrak{H}$:

$$\omega_f := f(z)(X - zY)^k dz. \tag{85}$$

Fix a point $z_0 \in \mathfrak{H}^*$. We may attach a cohomology class $\mathrm{ES}_k(f) \in H^1_p(\Gamma_0(M), L(k, \varepsilon; \mathbb{C}))$ defined by

$$\mathrm{ES}_k(f)(\gamma) := \int_{z_0}^{\gamma z_0} \omega_f \tag{86}$$

for each $\gamma \in \Gamma_0(M)$. The integral is independent of the choice of the point z_0. For either choice of sign $\pm$, we have a Hecke equivariant isomorphism

$$\mathrm{ES}^\pm_k : S_{k+2}(M, \varepsilon) \xrightarrow{\sim} H^1_p(\Gamma_0(M), L(k, \varepsilon; \mathbb{C}))^\pm; f \mapsto \mathrm{ES}^\pm_k(f) := \mathrm{ES}_k(f)^\pm \tag{87}$$

The additive map

$$\Phi_f : \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to L(k,\varepsilon;\mathbb{C}) \ ; \ \{c_2\} - \{c_1\} \mapsto \int_{c_1}^{c_2} \omega_f \tag{88}$$

defines a modular symbol in $\mathrm{Symb}_{\Gamma_0(M)}(L(k,\varepsilon;\mathbb{C}))$. Then $\mathrm{ES}_k^{\pm}(f)$ is the image of Φ_f under (79). Moreover, the map

$$S_{k+2}(M,\varepsilon) \to \mathrm{Symb}_{\Gamma_0(M)}(L(k,\varepsilon;\mathbb{C})) \ ; \ f \mapsto \Phi_f \tag{89}$$

is Hecke equivariant. Hence, by the Hecke equivariance of the Eichler-Shimura isomorphism (87), we see that for either choice of sign $\pm$,

$$s_{k,\varepsilon}\left(\mathrm{ES}_k^{\pm}(f)\right) = \Phi_f^{\pm}. \tag{90}$$

3.2. The cohomological D-th Shintani lifting

Let k be a non-negative integer, M an odd positive integer, χ a Dirichlet character modulo M, and D a fundamental discriminant with $\chi(-1)(-1)^{k+1}D > 0$. For each $Q \in \mathcal{L}_M(\Delta)$ with a positive integer Δ with $\Delta \equiv 0, 1 \pmod 4$, let $\partial C_Q \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the boundary of C_Q given by

$$\partial C_Q := \{\omega'_Q\} - \{\omega_Q\}, \tag{91}$$

where recall that (ω_Q, ω'_Q) is defined by (17) and that C_Q is the geodesic path oriented from ω_Q to ω'_Q. Let R be a commutative $\mathbb{Z}[\chi][\chi(-1)^{1/2}|D|^{1/2}G(\chi_0^{-1})]$-algebra such that $(2k)!$ is invertible in R.

Definition 3.1. 1. For each $\Phi \in \mathrm{Symb}_{\Gamma_0(M)}(L(2k,\chi^2;R))$ and each $Q \in \mathcal{L}_M(\Delta)$, we set

$$J_{k,\chi}(\Phi, Q) := \chi_0(Q) \cdot \left[\Phi(\partial C_Q), Q^k\right] \in R, \tag{92}$$

$$\gamma^M_{k,\chi,D}(\Phi; n, t) := \sum_{Q \in \mathcal{L}_{tc_\chi M}(\Delta_{n,t})/\Gamma_0(M)} \omega_D(Q) J_{k,\chi}(\Phi, Q) \tag{93}$$

2. For $\Phi \in \mathrm{Symb}_{\Gamma_0(M)}(L(2k,\chi^2;R))$, we define the n-th coefficient of $\Theta^M_{k,\chi,D}(\Phi) \in R[[q]]$ by

$$a_n(\Theta^M_{k,\chi,D}(\Phi)) := c_{k,\chi,D} \sum_{t | c_\chi^{-1} M} \mu\chi_D\chi_0^{-1}(t) t^{-k-1} \gamma^M_{k,\chi,D}(\Phi; n, t) \tag{94}$$

if $\chi(-1)(-1)^{k+1}n \equiv 0, 1 \pmod 4$ and $a_n(\Theta^M_{k,\chi,D}(\Phi)) := 0$ otherwise. Here, recall that $c_{k,\chi,D}$ and $\Delta_{n,t}$ is defined by (20) and (21), respectively.

Proposition 3.2. 1. *For any $\Phi \in \mathrm{Symb}_{\Gamma_0(M)}(L(2k,\chi^2;R))$, we have*

$$\Theta^M_{k,\chi,D}(\Phi|\iota) = -\Theta^M_{k,\chi,D}(\Phi). \tag{95}$$

2. *For any $f \in S_{2k+2}(M,\chi^2)$, we have*

$$\Theta^M_{k,\chi,D}(\Phi_f) = \Theta^M_{k,\chi,D}(\Phi_f^-) = \theta^M_{k,\chi,D}(f). \tag{96}$$

3. *If K is a field of characteristic zero and Φ belongs to the image of s_{2k,χ^2}, then*

$$\Theta^M_{k,\chi,D}(\Phi) \in S^+_{k+3/2}(4M, \tilde{\chi}; K). \tag{97}$$

Proof. The proof is essentially the same as [29, Proposition 4.3.3].

Let $f \in S_{2k+2}(M, \chi^2)$ be a Hecke eigenform, K the p-adic completion of the field obtained by adjoining the values of χ and $\chi(-1)^{1/2}|D|^{1/2}G(\chi_0^{-1})$ to the Hecke field $\mathbb{Q}_f$, and λ_f the $\mathcal{O}_K$-algebra homomorphism corresponding to f. By [13, Proposition 3.3], the eigenmodule $\mathrm{Symb}_{\Gamma_0(M)}(L(k,\chi^2;\mathcal{O}_K))^{\pm}[\lambda_f]$ is free of rank one over $\mathcal{O}_K$. Let $\Delta_f^{\pm}$ be a generator of $\mathrm{Symb}_{\Gamma_0(M)}(L(k,\chi^2;\mathcal{O}_K))^{\pm}[\lambda_f]$. This fact implies that there exists $\Omega(f)^{\pm} \in \mathbb{C}_p^{\times}$ such that

$$\Delta_f^{\pm} = (\Omega(f)^{\pm})^{-1} \cdot \Phi_f^{\pm} \in \mathrm{Symb}_{\Gamma_0(M)}(L(k,\chi^2;\mathcal{O}_K))^{\pm}[\lambda_f]. \tag{98}$$

Theorem 3.3. *Let $f \in S_{2k+2}(Np, \chi^2)$ be a Hecke eigenform with $\chi_D\chi(-1)(-1)^k = -1$. Then,*

$$(\Omega(f)^{-})^{-1} \cdot \theta_{k,\chi,D}^{Np}(f) = \Theta_{k,\chi,D}(\Delta_f^{-}) \in S_{k+3/2}^{+}(4Np, \tilde{\chi}; p^{-1}\mathcal{O}_K). \tag{99}$$

Proof. Since $\Delta_f^{-} \in \mathrm{Symb}_{\Gamma_0(Np)}(L(k,\varepsilon;\mathcal{O}_K))^{-}[\lambda_f]$, we have

$$\chi_0(Q) \cdot \left[\Delta_f^{-}(\partial C_Q), Q^k\right] \in \mathcal{O}_K. \tag{100}$$

The assertion follows from Proposition 3.2.

For a Hecke eigenform $f \in S_{2k+2}(Np, \chi^2)$ with $\chi_D\chi(-1)(-1)^k = -1$. We fix, once and for all, the complex period $\Omega(f)^{-}$ as (98) and define

$$\theta_D^{\mathrm{alg}}(f) := (\Omega(f)^{-})^{-1} p \cdot \theta_{k,\chi,D}^{Np}(f) \in \mathcal{O}_K[[q]]. \tag{101}$$

4. Rigid analytic ingredients

Let K be a complete discrete valuation field. The *weight space* $\mathcal{W}$ attached to $\mathcal{O}_K[\![\mathbb{Z}_p^{\times}]\!]$ is the rigid analytic variety whose $\mathbb{C}_p$-valued points are given by

$$\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^{\times}, \mathbb{C}_p^{\times}) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_K[\![\mathbb{Z}_p^{\times}]\!], \mathbb{C}_p). \tag{102}$$

For a K-Banach algebra R and an R-valued point $k \in \mathcal{W}(R)$, we will use a notation t^k instead of $k(t)$ for $t \in \mathbb{Z}_p^{\times}$. For a K-rigid analytic variety X, we denote by $A(X)$ the ring of rigid analytic functions on X and $A^{\circ}(X)$ the subring consisting of elements that are power bounded with respect to the supremum semi-norm $|\ |$ (see [4, Definition 6.2.1/2]). By [4, Proposition 6.2.3/1], we have $A^{\circ}(X) = \{f \in A(X) \mid |f| \leq 1\}$.

4.1. Coleman families

In this subsection, we recall Coleman families given in [7] following [33]. Let K be a complete subfield of $\mathbb{C}_p$ and $f \in S_{k_0}(Np, \varepsilon; K)_{\alpha}$ a Hecke eigenform with $k_0 - 1 > \alpha$. Assume that f is (p)-*new*, i.e., the primitive form attached to f is a newform of level either N or Np. We denote by ε_p the restriction of ε to $(\mathbb{Z}/p\mathbb{Z})^{\times}$. Then there exists an integer $0 \leq i \leq p-1$ such that we have $\varepsilon_p = \tau^{i-k_0}$, where $\tau : (\mathbb{Z}/p\mathbb{Z})^{\times} \hookrightarrow \mathbb{Z}_p^{\times}$ is the Teichmüller character. Let $T(n)$ be a Hecke operator on overconvergent forms defined in [7, Lemma B5.1 and p.464] for each positive integer n. Note that $T(n)$ coincides with the usual Hecke operator T_n on classical modular forms Let $S(N,i)$ be the K-vector space of families of cuspidal overconvergent forms of tame level N and type i defined in [7, Section B4]. Then by [7, Theorem B3.4], there exists a sufficiently large integer $m > (2-p)/(p-1)$ depending on α such that we can obtain a certain direct summand $S_B(N,i)_{\alpha}$ of the restriction of $S(N,i)$ on the affinoid disc $B = B_K[k_0, p^{-m}]$ of radius p^{-m} around k_0 defined over K, which interpolates the K-vector spaces $S_k^{\mathrm{cl}}(\omega^{i-k}; K)_{\alpha}$ of classical cusp forms of level Np, $(\mathbb{Z}/p\mathbb{Z})^{\times}$-character τ^{i-k} and $T(p)$-slope α with varying integral weights $k \in B(\mathbb{Z}) := B(\mathbb{C}_p) \cap \mathbb{Z} = \{k \in \mathbb{Z} \mid k \equiv k_0 \pmod{p^m}\}$ greater than $\alpha + 1$. Here the classicality of overconvergent forms of small $T(p)$-slope is given by [6, Theorem 6.1]. (Note that p^{-m} and $S_B(N,i)_\alpha$ are written as r and H in [7, the subsection "R-families" on the page 465], respectively.) The set of $\mathbb{C}_p$-valued points of B is given by

$$B(\mathbb{C}_p) = \{s \in \mathcal{O}_{\mathbb{C}_p} \mid |k_0 - s|_p \le p^{-m}\}. \tag{103}$$

The K-affinoid algebra $A(B)$ attached to B is the K-algebra $K\left\langle (X-k_0)/p^m \right\rangle$ of strictly convergent power series in $(X-k_0)/p^m$ with the indeterminate X (see [4, Proposition 6.1.4/4]). By [7, Theorem B3.4], we know that

$$\dim_K(S^{\mathrm{cl}}_{k_0}(\tau^{i-k_0}; K)_\alpha) = \dim_K(S^{\mathrm{cl}}_{k}(\tau^{i-k}; K)_\alpha) =: d \tag{104}$$

for all k in

$$W_B := \{k \in B(\mathbb{Z}) \mid k \equiv k_0 \pmod{p-1}, k > \alpha + 1\}. \tag{105}$$

Then we see that $S_B(N,i)_\alpha$ is a projective $A(B)$-module of rank d by [7, Theorem A4.5], and for any $k \in W_B$, we have the specialization map

$$\mathrm{sp}_k : S_B(N,i)_\alpha \twoheadrightarrow S_B(N,i)_\alpha \otimes_{A(B)} A(B)/P_k \xrightarrow{\sim} S^{\mathrm{cl}}_k(\tau^{i-k}; K)_\alpha, \tag{106}$$

where $P_k := (X-k)$ is the maximal ideal of $A(B)$. For any $k \in W_B$, we have $\tau^{i-k} = \tau^{i-k_0} = \varepsilon_p$. The (p)-new subspace $S^{(p)\text{-new}}_B(N,i)_\alpha$ of $S_B(N,i)_\alpha$ is defined as the intersection of kernels of all the degeneracy trace maps from level $\Gamma_1(Np)$ to level $\Gamma_1(N'p)$ for all positive divisors N' of N with $N' \neq N$. For any $k \in W_B$, we define the (p)-new subspace $S^{(p)\text{-new}}_k(\tau^{i-k}; K)_\alpha$ of $S^{\mathrm{cl}}_k(\tau^{i-k}; K)_\alpha$ as well. Then, we have the canonical isomorphism

$$S^{(p)\text{-new}}_B(N,i)_\alpha \otimes_{A(B)} A(B)/P_k \cong S^{(p)\text{-new}}_k(\tau^{i-k}; K)_\alpha \tag{107}$$

of finite dimensional K-vector spaces (see [33, Proposition 2.1]).

Definition 4.1. We define the subspace $S^{\mathrm{ss}}_k(K)$ of $S^{(p)\text{-new}}_k(\mathbb{1}_p; K)_\alpha$ as the subspaces spanned by primitive forms of level Np and character ε and old forms g and $g|V_p$ coming from primitive forms g of level N and character ε such that the characteristic polynomial of $T(p)$ acting on the subspaces spanned by g and $g|V_p$ has no double roots (see [33, Definition 2.2]).

Assume that $i \equiv k_0 \pmod{p-1}$. By (107), we have the specialization map

$$\mathrm{sp}_k : S^{(p)\text{-new}}_B(N,i)_\alpha \twoheadrightarrow S^{(p)\text{-new}}_B(N,i)_\alpha \otimes_{A(B)} A(B)/P_k \xrightarrow{\sim} S^{(p)\text{-new}}_k(\mathbb{1}_p; K)_\alpha \tag{108}$$

for any $k \in W_B$. Then we put

$$S^{\mathrm{ss}}_B := \mathrm{sp}_{k_0}^{-1}(S^{\mathrm{ss}}_{k_0}(K)) \subset S^{(p)\text{-new}}_B(N,i)_\alpha. \tag{109}$$

Definition 4.2. Let $\mathcal{H}_B$ be the Hecke algebra defined as the $A(B)$-subalgebra of $\mathrm{End}_{A(B)}(S_B(N,i)_\alpha)$ generated by Hecke operators $T(n)$ with all $n \ge 1$. We denote by $\mathcal{H}^{(p)\text{-new}}_B$ the image of the natural homomorphism

$$\mathcal{H}_B \to \mathrm{End}_{A(B)}(S^{(p)\text{-new}}_B(N,i)_\alpha) \tag{110}$$

given by the restricting the Hecke action. Since the $A(B)$-submodule S^{ss}_B defined by (109) is stable under the action of $\mathcal{H}^{(p)\text{-new}}_B$, we can take the image $\mathfrak{h}_B$ of the natural homomorphism

$$\mathcal{H}^{(p)\text{-new}}_B \to \mathrm{End}_{A(B)}(S^{\mathrm{ss}}_B) \tag{111}$$

given by restricting the Hecke action.

Then $\mathfrak{h}_B$ is a K-affinoid algebra which is finite over $A(B)$. We specialize $\mathfrak{h}_B$ at the closed point k_0 of B as $\mathfrak{h}_B \otimes_{A(B)} A(B)/P_{k_0}$ and take the image $\mathfrak{h}_{k_0}(K)$ of the natural homomorphism

$$\mathfrak{h}_B \otimes_{A(B)} A(B)/P_{k_0} \to \mathrm{End}_K(\mathrm{sp}_{k_0}(S_B^{\mathrm{ss}})) = \mathrm{End}_K(S_{k_0}^{\mathrm{ss}}(K)). \tag{112}$$

Then the Hecke algebra $\mathfrak{h}_{k_0}(K)$ is a commutative semi-simple K-algebra by the theory of newforms and old forms (see [19, Theorem 1]). By the definition of $\mathfrak{h}_B$ and $\mathfrak{h}_{k_0}(K)$, we have the natural surjective $A(B)$-algebra homomorphism

$$\mathfrak{sp}_{k_0} : \mathfrak{h}_B \twoheadrightarrow \mathfrak{h}_{k_0}(K). \tag{113}$$

Let $\lambda_1, \ldots, \lambda_r : \mathfrak{h}_{k_0}(K) \to K$ be the K-algebra homomorphisms which correspond to all Hekce eigenforms in $S_{k_0}^{\mathrm{ss}}(K)$ via the duality between classical Hecke eigenforms and K-algebra homomorphisms from a classical Hecke algebra into K (see [11, Proposition 3.21]) with some positive integer $r \leq d$. Let $\mathfrak{h}_B^{\mathrm{red}} := \mathfrak{h}_B/\sqrt{(0)}$ be the reduction of $\mathfrak{h}_B$. Since $\mathfrak{h}_{k_0}(K)$ is reduced, we see that (113) factors through the surjective $A(B)$-algebra homomorphism $\mathfrak{sp}_{k_0} : \mathfrak{h}_B^{\mathrm{red}} \twoheadrightarrow \mathfrak{h}_{k_0}(K)$.

Theorem 4.3 ([33, Theorem 2.2]). *We have the following commutative diagram of $A(B)$-algebras*

$$\begin{array}{ccccc} \mathfrak{h}_B^{\mathrm{red}} & \xrightarrow{\sim} & A(B)^r & ; & T \longmapsto (A_1(T), A_2(T), \ldots, A_r(T)) \\ \downarrow{\scriptstyle \mathfrak{sp}_{k_0}} & & \downarrow{\scriptstyle \mathrm{mod}\ P_{k_0}} & & \\ \mathfrak{h}_{k_0}(K) & \xrightarrow{\sim} & K^r & ; & T \longmapsto (\lambda_1(T), \lambda_2(T), \ldots, \lambda_r(T)) \end{array} \tag{114}$$

after shrinking the disk B around the center k_0 if necessary.

Let $f \in S_{k_0}^{\mathrm{new}}(N, \varepsilon)_\alpha$ be a primitive form with $k_0 - 1 > \alpha$. Assume that $\alpha \neq (k_0 - 1)/2$. Then the characteristic polynomial of $T(p)$ acting on the subspace spanned by f and $f|V_p$ has no double roots. We can take the root $\alpha_p(f)$ of the polynomial satisfying $\mathrm{ord}_p(\alpha_p(f)) = \alpha$. The *$p$-stabilization* f^* of f is the eigenvector with eigenvalue $\alpha_p(f)$ of T_p on the subspace given by

$$f^* := f - \varepsilon(p)p^{k_0-1}\alpha_p(f)^{-1} \cdot f|V_p. \tag{115}$$

The p-stabilization f^* is the Hecke eigenform of level Np with the same eigenvalues as f outside p and $T(p)$-eigenvalue $a_p(f^*) = \alpha_p(f)$. Let K be the p-adic completion of the field $\mathbb{Q}_f(\alpha_p(f))$ obtained by adjoining $\alpha_p(f)$ to the Hecke field $\mathbb{Q}_f$ of f. Then $f^* \in S_{k_0}^{\mathrm{ss}}(K)$. Let $\lambda_{f^*} : \mathfrak{h}_{k_0}(K) \to K$ be the K-algebra homomorphism corresponding to f^* via the duality and $A_{f^*} : \mathfrak{h}_B^{\mathrm{red}} \to A(B)$ the $A(B)$-algebra homomorphism whose specialization at k_0 coincides with $\lambda_{f^*}(\mathfrak{sp}_{k_0}(T))$ for any $T \in \mathfrak{h}_B^{\mathrm{red}}$, obtained in the theorem above. For all positive integers n, we put $a_n(\mathbf{f}) := A_{f^*}(T(n))$ for short. Then the formal power series $\mathbf{f} = \sum_{n \geq 1} a_n(\mathbf{f})q^n \in A(B)[[q]]$ interpolates Hecke eigenforms of level Np and we have the following:

Theorem 4.4 ([33, Corollary 2.3]). *Let $f \in S_{k_0}^{\mathrm{new}}(N, \varepsilon)_\alpha$ be a primitive form with $k_0 - 1 > \alpha \neq (k_0 - 1)/2$, and K a complete subfield of $\mathbb{C}_p$ containing the p-adic completion of the Hecke field $\mathbb{Q}_{f^*}$. Then there exist a K-affinoid disk $B_f = B_K[k_0, p^{-m_f}]$ with a positive integer m_f and a formal power series $\mathbf{f} \in A^\circ(B_f)[[q]]$ such that for any $k \in W_f := B_f(\mathbb{Z}) \cap W_B$ except for at most one (we call this element an exceptional weight), there exists a primitive form $f_k \in S_k^{\mathrm{new}}(N, \varepsilon; \mathcal{O}_K)_\alpha$ satisfying the following conditions:*

1. $\mathbf{f}(k) = f_k^*$.
2. $\mathbf{f}(k_0) = f^*$ *(i.e., $f_{k_0} = f$).*
3. $\mathbf{f}(k_1) \in S_{k_1}^{\mathrm{new}}(Np, \varepsilon)_\alpha$ *is primitive if there exists an exceptional weight $k_1 \in W_f$*

In particular, then there exists an integer $m_0 \geq m_f$ such that for any integer $r > m_0$, we have

$$f_k^* \equiv f^* \pmod{p^{r-m_0}\mathcal{O}_K} \text{ if } k \equiv k_0 \pmod{(p-1)p^r}. \tag{116}$$

Remark 4.5. In order to obtain a disk B_f in the theorem above, we shrink the disk B if necessary so that the following properties hold:

1. Theorem 4.3 is applicable.
2. the coefficients $a_n(\mathbf{f})$ of $\mathbf{f}$ satisfy $|a_n(\mathbf{f})| \leq 1$, i.e., $\mathbf{f} \in A^\circ(B_f)$.
3. the specializations $\mathbf{f}(k)$ have the same character ε.

It is possible to shrink B so that we have (2) by [7, the proof of Lemma B5.3] and (3) by [5, Lemma 5.5]. Thus. we may take a disk B' as the intersection of disks satisfying (1), (2), and (3).

We refer to $\mathbf{f}$ as a *Coleman family* passing through f^* as well as $\{f_k^*\}_{k \in W_f}$ obtained in the theorem above for a primitive form f.

4.2. Analytic functions and distributions

Let $\mathcal{W}^*$ be the rigid subspace of $\mathcal{W}$ consisting of accessible weights, i.e., weights k such that for any $t \in \mathbb{Z}_p^\times$, $|k(t)^{p-1} - 1| < p^{-1/(p-1)}$. Let U be an open K-affinoid subvariety of $\mathcal{W}^*$. We define the *universal weight* $k_U \in \mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^\times, A^\circ(U)^\times)$ by $t^{k_U}(x) := t^x$ for all $x \in U(K)$. Let R° denote one of the complete regular local Noetherian rings O_K and $A^\circ(U)$. For $R := R^\circ \hat{\otimes}_{\mathcal{O}_K} K$, we let $k_R \in \mathcal{W}^*(R)$ be an element that requires $k_R = k_U$ if $R = A(U)$. We denote by $A(k_R; R^\circ)$ the R°-module consisting of functions $f : \mathbb{Z}_p \times \mathbb{Z}_p^\times \to R^\circ$ such that for all $t \in \mathbb{Z}_p^\times$ and $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p^\times$, we have $f(tx, ty) = t^{k_R} f(x, y)$ and $f(z, 1) \in R^\circ\langle z\rangle$. We denote by $A(k_R, \varepsilon; R^\circ)$ the $R^\circ[\Gamma_0(Np)]$-module $A(k_R; R^\circ)$ equipped with the ε-twisted action; we let $\gamma \in \Gamma_0(Np)$ act on $f \in A(k_R; R^\circ)$ by

$$(\gamma \cdot f)(x, y) = \varepsilon(\gamma) f((x, y)^t\gamma), \tag{117}$$

where $\varepsilon(\gamma)$ is the value of ε on the lower right entry of γ and we assume that the restriction of k_U and ε to $(\mathbb{Z}/p\mathbb{Z})^\times$ coincide. We set

$$D(k_R, \varepsilon; R^\circ) := \mathrm{Hom}_{R^\circ}^{\mathrm{cont}}(A(k_R, \varepsilon; R^\circ), R^\circ). \tag{118}$$

and endow $D(k_R, \varepsilon; R^\circ)$ with $\Gamma_0(Np)$-action by

$$(\mu|\gamma)(f) := \mu(\gamma \cdot f) \tag{119}$$

for $f \in A(k_R, \varepsilon; R^\circ)$. Now we have natural specialization maps

$$A(k_U, \varepsilon; A^\circ(U)) \to A(k, \varepsilon; \mathcal{O}_K); f \mapsto f_k, \tag{120}$$

$$\eta_k : D(k_U, \varepsilon; A^\circ(U)) \to D(k, \varepsilon; \mathcal{O}_K); \mu \mapsto \mu_k, \tag{121}$$

where $f_k(x, y) := f(x, y)(k)$ and $\mu_k(f) := \mu(f_U)(k)$ with $f_U(x, y) := y^{k_U} f(x/y, 1)$ for $f \in A(k, \varepsilon; \mathcal{O}_K)$. Let t_k be an element of $A^\circ(U)$ which vanishes with order 1 at k and nowhere else. Then we have canonical exact sequences of $A^\circ(U)[\Gamma_0(Np)]$-modules

$$0 \to A(k_U, \varepsilon; A^\circ(U)) \xrightarrow{t_k} A(k_U, \varepsilon; A^\circ(U)) \to A(k, \varepsilon; \mathcal{O}_K) \to 0, \tag{122}$$

$$0 \to D(k_U, \varepsilon; A^\circ(U)) \xrightarrow{t_k} D(k_U, \varepsilon; A^\circ(U)) \xrightarrow{\eta_k} D(k, \varepsilon; \mathcal{O}_K) \to 0 \tag{123}$$

(see [1, Proposition 3.11]). Identifying $L(k,\varepsilon;\mathcal{O}_K)=\langle X^k, X^{k-1}Y,\dots,Y^k\rangle$ with the $\mathcal{O}_K[\Gamma_0(Np)]$-submodule $\mathcal{P}(k,\varepsilon;\mathcal{O}_K):=\langle y^k, y^{k-1}x,\dots,x^k\rangle$ of $A(k,\varepsilon;\mathcal{O}_K)$, and dualizing $\mathcal{P}(k,\varepsilon;\mathcal{O}_K)\subset A(k,\varepsilon;\mathcal{O}_K)$ give a $K[\Gamma_0(Np)]$-homomorphism

$$\rho_k : D(k,\varepsilon;\mathcal{O}_K)\twoheadrightarrow L(k,\varepsilon;\mathcal{O}_K);\mu\mapsto\sum_{i=0}^{k}\mu(y^{k-i}x^i)X^{k-i}Y^i=\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}(yX-xY)^k d\mu(x,y). \tag{124}$$

We define the $A^\circ(U)[\Gamma_0(Np)]$-homomorphism ϕ_k° as

$$\phi_k^\circ : D(k_U,\varepsilon;A^\circ(\Omega))\xrightarrow{\eta_k}\!\!\!\!\twoheadrightarrow D(k,\varepsilon;\mathcal{O}_K)\xrightarrow{\rho_k}\!\!\!\!\twoheadrightarrow L(k,\varepsilon;\mathcal{O}_K). \tag{125}$$

We set $A(k_R,\varepsilon;R):=A(k_R,\varepsilon;R^\circ)\hat{\otimes}_{\mathcal{O}_K}K$ and $D(k_R,\varepsilon;R):=D(k_R,\varepsilon;R^\circ)\hat{\otimes}_{\mathcal{O}_K}K$. Finally, we define the $A(U)[\Gamma_0(Np)]$-homomorphism ϕ_k by

$$\phi_k ;=\phi_k^\circ\hat{\otimes}_{\mathcal{O}_K}K : D(k_U,\varepsilon;A(U))\twoheadrightarrow L(k,\varepsilon;K), \tag{126}$$

4.3. Slope $\le h$ decompositon

Definition 4.6 ([2, Definition 4.1.1, 4.6.3 and 4.6.1 and Lemma 4.6.4]). Let $K\subset\mathbb{C}_p$ be a complete subfield, A a commutative Noetherian K-Banach algebra with norm $|\cdot|_A$, A^{m} the group of multiplicative units in A with respect to $|\cdot|_A$, and H an A-module with $u\in\mathrm{End}_A(H)$. For a polynomial $Q\in A[T]$, we denote by

$$Q^*(T):=T^{\deg(Q)}Q(1/T). \tag{127}$$

Let $h\in\mathbb{Q}$ and $A[T]_{\le h}$ the set of polynomials $Q\in A[T]$ such that $Q^*(0)\in A^{\mathrm{m}}$ and the slopes of Q are less than or equal to h (see [2] for the definition of slopes of a power series). A *slope $\le h$ decomposition* of H with respect to u is an $A[u]$-module decomposition $H=H_{\le h}\oplus H_{>h}$ such that

1. $H_{\le h}=\bigcup_{Q\in A[T]_{\le h}}\mathrm{Ker}\,Q^*(u)$ is finitely generated as an A-module
2. $Q^*(u)|_{H_{>h}}\in\mathrm{Aut}_A(H_{>h})$ for any $Q\in A[T]_{\le h}$.

Theorem 4.7. *Let $h\in\mathbb{Q}_{\ge 0}$.*

1. *For any $\kappa\in\mathcal{W}(K)$, there exists an open K-affinoid subvariety U in $\mathcal{W}$ containing κ such that an $A(U)$-module $\mathrm{Symb}_{\Gamma_0(Np)}(D(k_U,\varepsilon;A(U)))^\pm$ admits a slope $\le h$ decomposition with respect to the Hecke operator T_p.*
2. *The following control theorem holds:*

$$\mathrm{Symb}_{\Gamma_0(Np)}(D(k_U,\varepsilon;A(U)))^\pm_{\le h}\otimes_{A(U)}A(U)/P_k\cong\mathrm{Symb}_{\Gamma_0(Np)}(D(k,\varepsilon;K))^\pm_{\le h}, \tag{128}$$

where P_k is the maximal ideal of $A(U)$ generated by t_k.

3. *If $k+1>h$, the epimorphism ρ_k (124) induces the $K[\Gamma_0(Np)]$-isomorphism*

$$\mathrm{Symb}_{\Gamma_0(Np)}(D(k,\varepsilon;K))^\pm_{\le h}\xrightarrow{\sim}\mathrm{Symb}_{\Gamma_0(Np)}(L(k,\varepsilon;K))^\pm_{\le h}. \tag{129}$$

Remark 4.8. The theorem above was quoted in [23] without proof (see [23, Theorem 4.6] for (1) and [23, Theorem 4.12] for (2) and (3)). For more details, we refer to [2] and [1, Section 3]. In addition, [24] is useful especially for the comparison theorem (3).

5. *p*-Adic interpolation of the *D*-th Shintani lifting

Let $f\in S^{\mathrm{new}}_{k_0+2}(N,\varepsilon)_\alpha$ be a primitive form with $k_0+1>\alpha\neq(k_0+1)/2$, and K the p-adic completion of the field obtained by adjoining $\chi(-1)^{1/2}|D|^{1/2}G(\chi_0^{-1})$ and the values of χ to the Hecke field $\mathbb{Q}_{f^*}$. By Theorem 4.4, there exists a K-affinoid disk B_f around k_0+2 and a Coleman family $\mathbf{f}\in A^\circ(B_f)[[q]]$ passing through f^*. By Theorem 4.7.(1), there exists an open K-affinoid subvariety U in $\mathcal{W}^*$ containing $(k_0+2,\mathbb{1}_p)$ such that an $A(U)$-module $\mathrm{Symb}_{\Gamma_0(Np)}(D(k_U,\varepsilon;A(U)))^\pm$ admits a slope $\le\alpha$ decomposition with respect to the Hecke operator T_p.

5.1. Overconvergent Hecke eigensymbols

Lemma 5.1. *Let K be a complete subfield of $\mathbb{C}_p$. Let $k, n \in \mathcal{O}_K$ and $m \in \mathbb{Q}_{\geq 0}$. Then,*

$$\sigma_n^+ : K\langle (X-k)/p^m \rangle \xrightarrow{\sim} K\langle (X-(k+n))/p^m \rangle ; X \mapsto X - n \tag{130}$$

is an isometric K-algebra isomorphism with respect to the supremum semi-norm. In particular, the pair of σ_n^+ and

$$^a\sigma_n^+ : B_K[k+n, p^{-m}] \xrightarrow{\sim} B_K[k, p^{-m}]; \mathfrak{m} \mapsto (\sigma_n^+)^{-1}(\mathfrak{m}) \tag{131}$$

gives an isomorphism as K-affinoid varieties.

Proof. We put $T_2 := K\langle X, Y\rangle$ for short. Let ϕ be the K-algebra endomorphism of T_2 defined by $\phi(X) = X - n$ and $\phi(Y) = Y$ (see [4, Corollary 5.1.3/5]). Since the endomorphism defined by $X \mapsto X + n$ and $Y \mapsto Y$ gives the inverse of ϕ, we see that $\phi \in \mathrm{Aut}_{K\text{-alg}}(T_2)$. Write $\mathfrak{a}$ for the principal ideal of T_2 generated by $X - k - p^m Y$, and hence $\phi(\mathfrak{a}) = (X - (k+n) - p^m Y)$. Then the natural projection $T_2 \twoheadrightarrow T_2/\phi(\mathfrak{a})$ composed with ϕ induces the K-algebra isomorphism σ_n^+ by [4, Proposition 6.1.4/4]. Since σ_n^+ is an integral monomorphism, it is isometric by [4, Proposition 6.2.2/1].

We put

$$B_\sigma = B_K[k_0, p^{-m}] := {}^a\sigma_2^+(B_f) \cap U, \; B := B_K[k_0+2, p^{-m}], \tag{132}$$
$$W_{B,\sigma} := \{k \in B_\sigma(\mathbb{Z}) \mid k \equiv k_0 \pmod{p-1}, \; k+1 > \alpha\}. \tag{133}$$

We denote by $\sigma := \sigma_2^+ : A(B_\sigma) \to A(B)$ the K-algebra isomorphism given by the lemma above. We let $S_{B,\sigma}^{(p)\text{-new}}(N,i)_\alpha$ denote $S_B^{(p)\text{-new}}(N,i)_\alpha$ viewed as an $A(B_\sigma)$-module via σ and $S_{B,\sigma}^{\mathrm{ss}}$ denote S_B^{ss} viewed as an $A(B_\sigma)$-submodule of $S_{B,\sigma}^{(p)\text{-new}}(N,i)_\alpha$. By (108), we have

$$\begin{aligned}\mathrm{sp}_{k,\sigma} : S_{B,\sigma}^{(p)\text{-new}}(N,i)_\alpha &\twoheadrightarrow S_{B,\sigma}^{(p)\text{-new}}(N,i)_\alpha \otimes_{A(B_\sigma)} A(B_\sigma)/P_k \\ &\xrightarrow{\sim} S_{B,\sigma}^{(p)\text{-new}}(N,i)_\alpha \otimes_{A(B)} A(B)_\sigma/P_{k+2} \xrightarrow{\sim} S_{k+2}^{(p)\text{-new}}(\mathbb{1}_p; K)_\alpha\end{aligned} \tag{134}$$

for any $k \in W_{B,\sigma}$. Let $\{\mathbf{f}_1, \dots, \mathbf{f}_r\}$ be a basis of $S_{B,\sigma}^{\mathrm{ss}}$ consisting of *Hecke eigenforms* given by

$$\mathbf{f}_i := \sum_{n \geq 1} A_i(T(n)) q^n \tag{135}$$

for the $A(B)$-algebra homomorphisms $A_i : \mathfrak{h}_B^{\mathrm{red}} \to A(B)$ obtained in Theorem 4.3. We may assume that $\mathbf{f}_i \in A^\circ(B)$ after shrinking B if necessary (Remark 4.5). For any $k \in W_{B,\sigma}$, we put

$$S_{B,\sigma}^{\mathrm{ss},\circ} := \bigoplus_{i=1}^r A^\circ(B)_\sigma \mathbf{f}_i, \quad S_{k+2}^{\mathrm{ss}}(\mathcal{O}_K) := \bigoplus_{i=1}^r \mathcal{O}_K \mathrm{sp}_{k,\sigma}(\mathbf{f}_i), \tag{136}$$

where $A^\circ(B)_\sigma$ denote the admissible $\mathcal{O}_K$-algebra $A^\circ(B)$ viewed as an $A^\circ(B_\sigma)$-algebra via $\sigma : A^\circ(B_\sigma) \to A^\circ(B)$. On the other hand, by Theorem 4.7, for any $k \in W_{B,\sigma}$, the surjective $A(B_\sigma)[\Gamma_0(Np)]$-homomorphism ϕ_k (126) induces the surjective Hecke equivariant $A(B_\sigma)$-homomorphism ϕ_k^*

$$\begin{aligned}\phi_k^* : \mathrm{Symb}_{\Gamma_0(Np)}(D(k_{B_\sigma}, \varepsilon; A(B_\sigma)))_{\leq\alpha}^- &\twoheadrightarrow \mathrm{Symb}_{\Gamma_0(Np)}(D(k_{B_\sigma}, \varepsilon; A(B_\sigma)))_{\leq\alpha}^- \otimes_{A(B_\sigma)} A(B_\sigma)/P_k \\ &\xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(Np)}(D(k, \varepsilon; K))_{\leq\alpha}^- \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(Np)}(L(k, \varepsilon; K))_{\leq\alpha}^-.\end{aligned} \tag{137}$$

By (123), we see that ϕ_k^* preserves the integral structure:

$$\phi_k^* : \mathrm{Symb}_{\Gamma_0(Np)}(D(k_{B_\sigma}, \varepsilon; A^\circ(B_\sigma)))^-_{\leq\alpha} \twoheadrightarrow \mathrm{Symb}_{\Gamma_0(Np)}(L(k, \varepsilon; \mathcal{O}_K))^-_{\leq\alpha}. \tag{138}$$

Since $S^{\mathrm{ss}}_{k_0+2}(\mathcal{O}_K)$ is spanned by Hecke eigenforms g of level Np, the $\mathcal{O}_K$-linear extension of the map $g \mapsto \Delta_g^-$ gives the injective Hecke equivariant $\mathcal{O}_K$-homomorphism

$$\xi_{k_0} : S^{\mathrm{ss}}_{k_0+2}(\mathcal{O}_K) \hookrightarrow \mathrm{Symb}_{\Gamma_0(Np)}(L(k_0, \varepsilon; \mathcal{O}_K))^-_{\leq\alpha}. \tag{139}$$

We put

$$\mathrm{Symb}^{\mathrm{ss}}_{k_0}(\mathcal{O}_K) := \xi_{k_0}(S^{\mathrm{ss}}_{k_0+2}(\mathcal{O}_K)), \quad \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma} := (\phi_{k_0}^*)^{-1}(\mathrm{Symb}^{\mathrm{ss}}_{k_0}(\mathcal{O}_K)). \tag{140}$$

Let $\mathfrak{h}^{\mathrm{red}}_{B,\sigma}$ denote $\mathfrak{h}^{\mathrm{red}}_B$ viewed as an $A^\circ(B_\sigma)$-algebra via $\sigma : A^\circ(B_\sigma) \to A^\circ(B)$. Let $\mathfrak{h}^{\mathrm{red},\circ}_{B,\sigma}$ be the $A^\circ(B)_\sigma$-subalgebra of $\mathfrak{h}^{\mathrm{red}}_{B,\sigma}$ generated by the Hecke eigensystems corresponding to the basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_r\}$ of $S^{\mathrm{ss}}_{B,\sigma}$ and $\mathfrak{h}_{k_0+2}(\mathcal{O}_K)$ the $\mathcal{O}_K$-subalgebra of $\mathfrak{h}_{k_0+2}(K)$ generated by the Hecke eigensystems corresponding to the basis $\{\mathrm{sp}_{k,\sigma}(\mathbf{f}_1), \ldots, \mathrm{sp}_{k,\sigma}(\mathbf{f}_r)\}$ of $S^{\mathrm{ss}}_{k+2}(\mathcal{O}_K)$. Then $\mathrm{Symb}^{\mathrm{ss}}_{k_0}(\mathcal{O}_K)$ (resp. $\mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma}$) is a module over $\mathfrak{h}_{k_0+2}(\mathcal{O}_K)$ (resp. $\mathfrak{h}^{\mathrm{red},\circ}_{B,\sigma}$) via the homomorphisms which send $T(\ell)$ to the usual Hecke operator T_ℓ.

Proposition 5.2. *There exists a $\mathfrak{h}^{\mathrm{red},\circ}_{B,\sigma}$-isomorphism $\Xi : S^{\mathrm{ss},\circ}_{B,\sigma} \xrightarrow{\sim} \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma}$ such that the following diagram commutes:*

$$\begin{array}{ccc} S^{\mathrm{ss},\circ}_{B,\sigma} & \xrightarrow{\ \Xi\ } & \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma} \\ \downarrow{\scriptstyle \mathrm{sp}_{k_0,\sigma}} & & \downarrow{\scriptstyle \phi^*_{k_0}} \\ S^{\mathrm{ss}}_{k_0+2}(\mathcal{O}_K) & \xrightarrow{\ \xi_{k_0}\ } & \mathrm{Symb}^{\mathrm{ss}}_{k_0}(\mathcal{O}_K) \end{array} \tag{141}$$

after shrinking the disk B_σ around the center k_0 if necessary.

Proof. We put $A := A^\circ(B_\sigma)$, $\mathfrak{h} := \mathfrak{h}^{\mathrm{red},\circ}_{B,\sigma}$, $S := S^{\mathrm{ss},\circ}_{B,\sigma}$, and $\mathrm{Symb} := \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma}$ for short. Let t_{k_0} be a generator of the maximal ideal P_{k_0} of A at the closed point k_0. Since ξ_{k_0} gives the isomorphism $S/t_{k_0}S \xrightarrow{\sim} \mathrm{Symb}/t_{k_0}\mathrm{Symb}$, it suffices to prove that there exists a $\mathfrak{h}$-isomorphism $\Xi : S \xrightarrow{\sim} \mathrm{Symb}$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\ \Xi\ } & \mathrm{Symb} \\ \downarrow{\scriptstyle \mathrm{sp}_{k_0,\sigma}} & & \downarrow{\scriptstyle \phi^*_{k_0}} \\ S/t_{k_0}S & \xrightarrow{\ \xi_{k_0}\ } & \mathrm{Symb}/t_{k_0}\mathrm{Symb} \end{array} \tag{142}$$

after shrinking the disk B_σ around the center k_0 if necessary. Let $\mathfrak{h}_{(k_0)} := \mathfrak{h} \otimes_A A_{P_{k_0}}$ be the localization of $\mathfrak{h}$ at P_{k_0}. Since $\mathfrak{h}_{(k_0)}$ is Noetherian and not Artinian, we see that the Krull dimension of $\mathfrak{h}$ is 1 by Krull's principal ideal theorem (see [18, Theorem 13.5]). By [18, Theorem 2.3], the embedding dimension of $\mathfrak{h}$ is 1, and hence $\mathfrak{h}$ is a regular local ring of Krull dimension 1. By [18, Theorem 19.2], the global dimension of $\mathfrak{h}$ is 1, which implies Symb has a finite injective dimension less than or equal to 1 by [18, Lemma 2, Section 19]. Let $S_{(k_0)} := S \otimes_{\mathfrak{h}} \mathfrak{h}_{(k_0)}$ and $\mathrm{Symb}_{(k_0)} := \mathrm{Symb} \otimes_{\mathfrak{h}} \mathfrak{h}_{(k_0)}$ be the localizations at P_{k_0}. Let $t_{(k_0)}$ be the image of t_{k_0} in $\mathfrak{h}_{(k_0)}$, and hence $t_{(k_0)}$ belongs to the annihilator of $\mathfrak{h}_{(k_0)}/P_{k_0}\mathfrak{h}_{(k_0)}$. Since $\mathfrak{h}_{(k_0)}$ is A-torsion-free and A is an integral domain, we see that $t_{(k_0)}$ is $\mathfrak{h}_{(k_0)}$-regular, $S_{(k_0)}$-regular, and $\mathrm{Symb}_{(k_0)}$-regular. By [18, Lemma 2, Section 18], we see that both $S_{(k_0)}$ and $\mathrm{Symb}_{(k_0)}$ are maximal Cohen-Macaulay modules. By [8, Proposition

21.13], there exists a $\mathfrak{h}_{(k_0)}$-isomorphism $\Xi_{(k_0)} : S_{(k_0)} \xrightarrow{\sim} \mathrm{Symb}_{(k_0)}$ such that the following diagram commutes:

$$\begin{array}{ccc} S_{(k_0)} & \xrightarrow{\Xi} & \mathrm{Symb}_{(k_0)} \\ {\scriptstyle \mathrm{sp}_{k_0,\sigma}} \downarrow & & \downarrow {\scriptstyle \phi^*_{k_0}} \\ S_{(k)}/t_k S_{(k_0)} & \xrightarrow{\xi_{k_0}} & \mathrm{Symb}_{(k_0)}/t_{k_0}\,\mathrm{Symb}_{(k_0)}\,. \end{array} \tag{143}$$

Therefore we obtain the desired commutative diagram after shrinking the disk B_σ around the center k_0 if necessary.

By the proposition above, we have the stronger result than [23, Theorem 4.13] in that we can take an error term (denoted by Ω_κ in [23]) of the p-adic interpolation as a p-adic unit u_k as follows:

Theorem 5.3. *Let $f \in S^{\mathrm{new}}_{k_0+2}(N,\varepsilon)_\alpha$ be a primitive form with $k_0+1 > \alpha \neq (k_0+1)/2$, K a complete subfield of $\mathbb{C}_p$ containing the p-adic completion of the Hecke field $\mathbb{Q}_{f^*}$, and $\mathbf{f}$ a Coleman family passing through f^*. Then there exist a K-affinoid disk $B = B_K[k_0, p^{-m}]$ with some positive integer m and a Hecke eigenvector $\Phi_{\mathbf{f}} \in \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma}$ with the same eigenvalues as $\mathbf{f}$ such that for any $k \in W_{B,\sigma}$, there exists $u_k \in \mathcal{O}_K^\times$ such that we have the following:*

1. $\phi^*_k(\Phi_{\mathbf{f}}) = u_k \Delta^-_{\mathbf{f}(k+2)}$.
2. $\phi^*_{k_0}(\Phi_{\mathbf{f}}) = \Delta^-_{f^*}$ *(i.e., $u_{k_0} = 1$).*

Proof. The Hecke equivariant isomorphism Ξ as Proposition 5.2 induces a Hecke equivariant $\mathcal{O}_K$-isomorphism Ξ_k as follows:

$$\begin{array}{ccc} S^{\mathrm{ss},\circ}_{B,\sigma} & \xrightarrow{\Xi} & \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma} \\ {\scriptstyle \mathrm{sp}_{k,\sigma}} \downarrow & & \downarrow {\scriptstyle \phi^*_{k}} \\ S^{\mathrm{ss}}_{k+2}(\mathcal{O}_K)_\alpha & \xrightarrow{\Xi_k} & \mathrm{Symb}^{\mathrm{ss}}_{k}(\mathcal{O}_K)_\alpha, \end{array} \tag{144}$$

We put $\Phi_{\mathbf{f}} := \Xi(\mathbf{f})$. Then we see that $\phi^*_k(\Phi_{\mathbf{f}}) = \Xi_k(\mathbf{f}(k+2))$ is a generator of $\lambda_{\mathbf{f}(k+2)}$-eigenmodule

$$\mathrm{Symb}_{\Gamma_0(Np)}(L(k,\varepsilon;\mathcal{O}_K))^-[\lambda_{\mathbf{f}(k+2)}]. \tag{145}$$

By [13, Proposition 3.3], the $\lambda_{\mathbf{f}(k+2)}$-eigenmodule is generated by $\Delta_{\mathbf{f}(k+2)}$ over $\mathcal{O}_K$. We thus the first assertion and the second assertion follows from $\Xi_{k_0} = \xi_{k_0}$.

We refer to $\Phi_{\mathbf{f}}$ obtained in the theorem above as a *Hecke eigensymbol* attached to a Coleman family $\mathbf{f}$.

5.2. A p-adic analytic family of the D-th Shintani lifting for a Coleman family

Hereafter, we assume that k_0 is even and $\varepsilon = \chi^2$ with a Dirichlet character χ modulo N. We replace the notation k_0 by $2k_0$ so that we remark that the set $W_{B,\sigma}$ defined by (133) is replaced as follows:

$$W_{B,\sigma} = \{k \in \mathbb{Z} \mid k \equiv 2k_0 \pmod{(p-1)p^m}, k+1 > \alpha\}\,. \tag{146}$$

We consider the family of $\theta^{\mathrm{alg}}_D(\mathbf{f}(2k+2))$'s for $2k \in W_{B,\sigma}$. Let n be a positive integer with $\chi(-1)(-1)^{k+1}n \equiv 0, 1 \pmod 4$. We define the n-th coefficient of a formal power series that interpolates the family of the D-th Shintani lifting below. Let t be a positive divisor of N/c_χ and $Q \in \mathcal{L}_{tc_\chi Np}(\Delta_{n,t})$. Assume that $\mathrm{ord}_p(n) \leq 1$. Then we have the following:

Lemma 5.4. *Let c be the integer given by $[a,b,c]=Q$. Then we have $p\nmid c$. In particular, for any $(x,y)\in\mathbb{Z}_p\times\mathbb{Z}_p^\times$, we have $Q(x,y)\in\mathbb{Z}_p^\times$.*

Proof. We put $\Delta:=\Delta_{n,t}$ for short. By (39), there exist a positive integer l with $l^2\mid\Delta$, a integer $\varrho\in S_{Np}(\Delta/l^2)$, and $m\parallel m(l,\varrho):=(Np,\varrho,(\varrho^2-\Delta/l^2)/4Np)$ such that $Q\in l\cdot\mathcal{L}^0_{Np,\varrho,m,m(l,\varrho)/m}(\Delta/l^2)$. Since $\Delta\not\equiv 0 \pmod{p^2}$ from $\mathrm{ord}_p(n)\le 1$, we have $p\nmid l$. If $p\mid m(l,\varrho)$, then we have $p\mid\varrho$ and $\varrho^2\equiv\Delta/l^2 \pmod{p^2}$, and hence $\Delta/l^2\equiv 0 \pmod{p^2}$. This is a contradiction to $\Delta\not\equiv 0\pmod{p^2}$. Thus we have $p\nmid m(l,\varrho)$, and hence $p\nmid c$.

By the lemma above, we see that $Q(x,y)^{k_{B_\sigma}}$ is well-defined analytic function on $\mathbb{Z}_p\times\mathbb{Z}_p^\times$. We define $J_Q\in\mathrm{Hom}_{A^\circ(B_\sigma)}(D(k_{B_\sigma},\chi^2;A^\circ(B_\sigma)),A^\circ(B_\sigma))$ by

$$J_Q(\mu):=\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}Q(x,y)^{k_{B_\sigma}}\,d\mu(x,y) \tag{147}$$

Then we have the following:

Lemma 5.5. *For any $2k\in W_{B,\sigma}$ and $\mu\in D(k_{B_\sigma},\chi^2;A^\circ(B_\sigma))$, we have*

$$J_Q(\mu)(2k)=[\phi_{2k}(\mu),Q^k(X,Y)]. \tag{148}$$

In particular, by Theorem 5.3, we have

$$\chi_0(Q)J_Q(\Phi_{\mathbf{f}}(\partial C_Q))(2k)=u_{2k}(\Omega(\mathbf{f}(2k+2))^-)^{-1}I_{k,\chi}(\mathbf{f}(2k+2),Q) \tag{149}$$

Proof.

$$\begin{aligned}J_Q(\mu)(2k)&=\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}Q(x,y)^k d\mu_{2k}(x,y)\\&=\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\left[(yX-xY)^{2k},Q^k(X,Y)\right]d\mu_{2k}(x,y)\\&=\left[\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}(yX-xY)^{2k}d\mu_{2k}(x,y),Q^k(X,Y)\right]=[\phi_{2k}(\mu),Q^k(X,Y)].\end{aligned}$$

Definition 5.6. Let D be a fundamental discriminant with $\chi(-1)(-1)^{k_0+1}D>0$ and $(D,Np)=1$, and $\Phi_{\mathbf{f}}\in\mathrm{Symb}^{\mathrm{ss}}_{B_\sigma}$ a Hecke eigensymbol attached to $\mathbf{f}$. Let n be a positive integer with $\chi(-1)(-1)^{k+1}n\equiv 0,1 \pmod 4$ and $\mathrm{ord}_p(n)\le 1$, t a positive divisor of N/c_χ, and $Q\in\mathcal{L}_{tc_\chi Np}(\Delta_{n,t})$. We set

$$J_{B_\sigma}(Q):=\chi_0(Q)J_Q(\Phi_{\mathbf{f}}(\partial C_Q))\in A^\circ(B_\sigma). \tag{150}$$

We put

$$a_n(\theta_{B_\sigma,D}(\mathbf{f})):=\sum_{t\mid c_\chi^{-1}N}\mu\chi_D\chi_0^{-1}(t)t^{-k_{B_\sigma}-1}\sum_{Q\in\mathcal{L}_{tc_\chi Np}(\Delta_{n,t})/\Gamma_0(Np)}\omega_D(Q)J_{B_\sigma}(Q). \tag{151}$$

Let m be a positive integer and v a non-negative integer such that $0\le\mathrm{ord}_p(m/p^{2v})\le 1$. We put

$$a_m(\theta_{B_\sigma,D}(\mathbf{f})):=a_p(\mathbf{f})^v a_{m/p^{2v}}(\theta_{B_\sigma,D}(\mathbf{f})) \tag{152}$$

if $\chi(-1)(-1)^{k+1}m\equiv 0,1 \pmod 4$ and $a_m(\theta_{B_\sigma,D}(\mathbf{f})):=0$ otherwise. For $i\in\mathbb{Z}/4\mathbb{Z}$, we define the n-th coefficient of $\theta^i_{B_\sigma,D}(\mathbf{f})\in A^\circ(B_\sigma)[[q]]$ by

$$a_n(\theta^i_{B_\sigma,D}(\mathbf{f})):=\left(1-p^{-1}\right)c^i_{B_\sigma,D}\cdot a_n(\theta_{B_\sigma,D}(\mathbf{f})), \tag{153}$$

where

$$c^i_{B_\sigma,D}:=(-1)^{[(i+1)/2]}\chi_D(c_\chi)\chi(-1)^{1/2}\chi^{-1}(D)2^{k_{B_\sigma}+1}c_\chi^{k_{B_\sigma}}G(\chi_0^{-1}). \tag{154}$$

We then have the main theorem as follows:

Theorem 5.7. *Let $f \in S^{\mathrm{new}}_{2k_0+2}(N, \chi^2)_\alpha$ be a primitive form with $2k_0+1 > \alpha \neq (2k_0+1)/2$ and $c_\chi \parallel N$, K the p-adic completion of the field obtained by adjoining $\chi(-1)^{1/2}|D|^{1/2}G(\chi_0^{-1})$ and the values of χ to the Hecke field $\mathbb{Q}_{f^*}$, D a fundamental discriminant with $\chi(-1)(-1)^{k_0+1}D > 0$ and $(D, Np) = 1$. Then there exists a positive integer m_0 such that for any $r > m_0 + 1$, if an integer k satisfies $2k+1 > \alpha$ and $2k \equiv 2k_0 \pmod{(p-1)p^r}$, then there exist a primitive form $f_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N, \chi^2; \mathcal{O}_K)_\alpha$ such that*

$$e_k \theta^{\mathrm{alg}}_D(f^*_{2k+2}) \equiv \theta^{\mathrm{alg}}_D(f^*) \pmod{p^{r-m_0}\mathcal{O}_K} \tag{155}$$

for some $e_k \in \mathcal{O}_K^\times$ and f^*_{2k+2} lies in a Coleman family passing through f^*.

Proof. By Theorem 5.3, we have $\Phi_{\mathbf{f}} \in \mathrm{Symb}^{\mathrm{ss},\circ}_{B_\sigma}$ such that for any $2k \in W_{B,\sigma}$, there exists $u_{2k} \in \mathcal{O}_K^\times$ such that $\phi^*_{2k}(\Phi_{\mathbf{f}}) = u_{2k}\Delta_{\mathbf{f}(2k+2)}$ and $u_{2k_0} = 1$. Recall that $\mathbf{f}(2k+2) = f^*_{2k+2}$ for a primitive form $f_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N, \chi^2; \mathcal{O}_K)_\alpha$ by Theorem 4.4. Set $e_k := (-1)^{[(k_0+1)/2]}(-1)^{[(k+1)/2]}u_{2k}$ By Theorem 3.3 and Lemma 5.5, we see that $p \cdot \theta^{k_0}_{B_\sigma,D}(\mathbf{f}) \in A^\circ(B_\sigma)$ has the specialization $\theta^{k_0}_{B_\sigma,D}(\mathbf{f})(2k) = e_k\theta^{\mathrm{alg}}_D(f^*_{2k+2}) \in S^+_{k+3/2}(4Np, \tilde{\chi}; \mathcal{O}_K)$.

Remark 5.8. The p-adic interpolation of the classical Shintani lifting has already been done by Stevens [29] and Park [23] for a Hida family and a Coleman family, respectively. Roughly speking, Park proved that for all $n \geq 1$,

$$\left|\Omega_k \cdot a_n(\theta^{\mathrm{alg}}_1(f^*_{2k+2})) - a_n(\theta^{\mathrm{alg}}_1(f^*))\right|_p < 1 \tag{156}$$

for some $\Omega_k \in K^\times$ in [23]. The significant difference between their results and our result above is that we can take the error term e_k of the p-adic interpolation as a p-adic unit, and hence the congruence makes sense. Indeed, on the congruence (155), we see that $a_n(\theta^{\mathrm{alg}}_D(f^*_{2k+2}))$ vanishes modulo p if and only if $a_n(\theta^{\mathrm{alg}}_D(f^*))$ vanishes modulo p. However, even if we assume $\Omega_k \in \mathcal{O}_K$ on (156), the congruence

$$\Omega_k \cdot a_n(\theta^{\mathrm{alg}}_1(f^*_{2k+2})) \equiv a_n(\theta^{\mathrm{alg}}_1(f^*)) \pmod{p^{r-m_0}\mathcal{O}_K} \tag{157}$$

cannot tell us that $\theta^{\mathrm{alg}}_1(f^*_{2k+2})$ vanish modulo p if $\theta^{\mathrm{alg}}_1(f^*)$ vanish modulo p unless Ω_k is a p-adic unit.

We keep the notation as in the theorem above. Since $f_{2k+2} \otimes \chi_D\chi_0^{-1}$ and $f^*_{2k+2} \otimes \chi_D\chi_0^{-1}$ are Hecke eigenforms of trivial character ([20, Lemma 4.3.10]), we have

$$L\left(k+1, f^*_{2k+2} \otimes \chi_D\chi_0^{-1}\right) = \left(1 - \chi_D\chi^{-1}(p)p^k a_p(f^*_{2k+2})^{-1}\right) L\left(k+1, f_{2k+2} \otimes \chi_D\chi_0^{-1}\right) \tag{158}$$

by [20, Theorem 4.5.16]. We put

$$L^{\mathrm{alg}}\left(k+1, f^*_{2k+2} \otimes \chi_D\chi_0^{-1}\right) := \frac{k!L\left(k+1, f^*_{2k+2} \otimes \chi_D\chi_0^{-1}\right)}{\pi^{k+1}\Omega(f^*_{2k+2})^-} \in \mathcal{O}_K. \tag{159}$$

Then by Proposition 2.10 and Theorem 2.4, we have

$$e_k^{-1}a_{|D|}(\theta^{k_0}_{B_\sigma,D}(\mathbf{f}))(2k) = (\Omega(f^*_{2k+2})^-)^{-1}a_{|D|}\left(\theta^{Np}_{k,\chi,D}(f^*_{2k+2})\right) \tag{160}$$

$$= 2\left(1-p^{-1}\right)|D|^{k+1/2}c_\chi^{2k+1}R_D(f_{2k+2})L^{\mathrm{alg}}\left(k+1, f^*_{2k+2} \otimes \chi_D\chi_0^{-1}\right). \tag{161}$$

Since $2\left(1-p^{-1}\right)|D|^{k_B+1/2}N^{2k_B+1} \in A(B_\sigma)^\times$, we can normalize $a_{|D|}(\theta^{k_0}_{B_\sigma,D}(\mathbf{f}))$ as

$$L_D(\mathbf{f}) := \left(2\left(1-p^{-1}\right)|D|^{k_B+1/2}c_\chi^{2k_B+1}\right)^{-1} a_{|D|}(\theta^{k_0}_{B_\sigma,D}(\mathbf{f})) \in A(B_\sigma) \tag{162}$$

so that for any $2k \in W_{B,\sigma}$, we have

$$e_k^{-1} L_D(\mathbf{f})(2k) = R_D(f_{2k+2}) L^{\mathrm{alg}}\left(k+1, f_{2k+2}^* \otimes \chi_D \chi_0^{-1}\right). \tag{163}$$

Corollary 5.9. *Let the notation and the assumptions be the same as Theorem 5.7. Then there exists a positive integer r such that for any integer k satisfying $2k+1 > \alpha$ and $2k \equiv 2k_0 \pmod{(p-1)p^r}$, we have the following non-negative equality:*

$$\mathrm{ord}_p\left(R_D(f_{2k+2}) L^{\mathrm{alg}}\left(k+1, f_{2k+2}^* \otimes \chi_D \chi_0^{-1}\right)\right) = \mathrm{ord}_p\left(R_D(f) L^{\mathrm{alg}}\left(k_0+1, f^* \otimes \chi_D \chi_0^{-1}\right)\right) \tag{164}$$

Moreover, if $R_D(f) L\left(k_0+1, f \otimes \chi_D \chi_0^{-1}\right) \neq 0$, then we have

$$\mathrm{ord}_p\left(L^{\mathrm{alg}}\left(k+1, f_{2k+2}^* \otimes \chi_D \chi_0^{-1}\right)\right) = \mathrm{ord}_p\left(L^{\mathrm{alg}}\left(k_0+1, f^* \otimes \chi_D \chi_0^{-1}\right)\right) \geq 0, \tag{165}$$

in particular, $L\left(k+1, f_{2k+2} \otimes \chi_D \chi_0^{-1}\right) \neq 0$.

Proof. By Theorem 5.7, there exists a positive integer m_0 such that for any $r > m_0+1$, if an integer k satisfies $2k+1 > \alpha$ and $2k \equiv 2k_0 \pmod{(p-1)p^r}$, then

$$e_k R_D(f_{2k+2}) L^{\mathrm{alg}}\left(k+1, f_{2k+2}^* \otimes \chi_D \chi_0^{-1}\right) \equiv R_D(f) L^{\mathrm{alg}}\left(k_0+1, f^* \otimes \chi_D \chi_0^{-1}\right) \pmod{p^{r-m_0} \mathcal{O}_K} \tag{166}$$

for some $e_k \in \mathcal{O}_K$. Taking sufficiently large r, we have the first assertion. The last assertion follows from Remark 2.5.(1)

Remark 5.10. We keep the notation as in the corollary above. In general, $R_D(f_{2k+2})$ may vanish. However, as seen in the proof above, if $R_D(f) L\left(k_0+1, f \otimes \chi_D \chi_0^{-1}\right) \neq 0$, then $R_D(f_{2k+2}) \neq 0$ in a neighborhood of k_0. In other words, the the signatures of the eigenvalues of the initial primitive form f for the Atkin-Lehner involutions coincide with that of f_{2k+2} for k sufficiently close to k_0, p-adically (see Remark 2.5.(3)).

6. Application

We apply Corollary 5.9 assuming that $\chi = \mathbb{1}$, $\alpha = 0$, and N is square-free.

6.1. Congruences between the central L-values attached to cusp forms of different weights

Theorem 6.1. *Let $f \in S_{2k+2}^{\mathrm{new}}(N, \mathbb{1})_0$ and $g \in S_{2k'+2}^{\mathrm{new}}(N, \mathbb{1})_0$ be primitive forms with $k, k' \geq 0$, and $\mathcal{O}$ the ring of integers of the p-adic completion of the field obtained by adjoining $G(\chi_D)$ to the composite field $\mathbb{Q}_{f^*}\mathbb{Q}_{g^*}$. Assume that $f^* \equiv g^* \pmod{p^{r_0}\mathcal{O}}$ for some positive integer r_0 and that $k \equiv k' \pmod{(p-1)p^r}$ for a sufficiently large integer r and that the Galois representation $\rho_{f^*} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ attached to f^* is residually irreducible. Let D be a fundamental discriminant with $(-1)^{k+1}D > 0$ and $(D, Np) = 1$. Then there exist $e_{k'} \in \mathcal{O}^\times$ such that we have*

$$R_D(f) L^{\mathrm{alg}}\left(k+1, f^* \otimes \chi_D\right) \equiv e_{k'} R_D(f_{2k'+2}) L^{\mathrm{alg}}\left(k'+1, g^* \otimes \chi_D\right) \pmod{p^{r_0}\mathcal{O}}. \tag{167}$$

Moreover, if $R_D(f) L\left(k+1, f \otimes \chi_D\right) \neq 0$, then we have

$$L^{\mathrm{alg}}\left(k+1, f^* \otimes \chi_D\right) \equiv e_{k'} L^{\mathrm{alg}}\left(k'+1, g^* \otimes \chi_D\right) \pmod{p^{r_0}\mathcal{O}}. \tag{168}$$

Remark 6.2. When $k = k'$ in the theorem above, we can take $e_{k'} = 1$ by [30, Corollary 1.11]. Namely, the result in this case is contained in [30, Corollary 1.11].

Proof. Since f is p-ordinary and ρ_{f^*} is residually irreducible, we may identify our periods defined by (98) with canonical periods in the sense of [30] by [30, Theorem 1.13] and [31, Lemma 3.8]. By Theorem 4.4 and Corollary 5.9, we have

$$f^*_{2k'+2} \equiv f^* \equiv g^* \pmod{p^{r_0}\mathcal{O}}, \tag{169}$$

$$e_{k'} R_D(f_{2k'+2}) L^{\mathrm{alg}}\left(k'+1, f^*_{2k'+2} \otimes \chi_D\right) \equiv R_D(f) L^{\mathrm{alg}}\left(k+1, f^* \otimes \chi_D\right) \pmod{p^{r_0}\mathcal{O}} \tag{170}$$

for some $e_{k',D} \in \mathcal{O}^\times$. By [30, Corollary 1.11], the congruence (169) between $f^*_{2k'+2}$ and g^* implies

$$L^{\mathrm{alg}}\left(k'+1, f^*_{2k'+2} \otimes \chi_D\right) \equiv L^{\mathrm{alg}}\left(k'+1, g^* \otimes \chi_D\right) \pmod{p^{r_0}\mathcal{O}}. \tag{171}$$

6.2. The Goldfeld conjecture

We first recall Vatsal's result on the Goldfeld conjecture in [30, Section 3]. Let E be an elliptic curve over $\mathbb{Q}$ with a rational potint of order 3. Assume that E has good ordinary reduction at 3 and that the conductor N of E is square-free. Let $q \nmid N$ be any odd prime with $q \equiv 1 \pmod 9$. Let N_1 be the product of primes $\ell | N$ at which E has nonsplit multiplicative reduction and $N_2 := qN/N_1$. We denote by $f_E \in S_2^{\mathrm{new}}(N, \mathbb{1})_0$ the 3-ordinary primitive form attached to E and f_E^q its q-stabilization.

Theorem 6.3 ([30, Theorem 3.3]). *For any negative fundamental discriminant D with $(D, Nq) = 1$, we have the congruence*

$$L^{\mathrm{alg}}\left(1, f_E^q \otimes \chi_D\right) \equiv \frac{1}{2} \prod_{\ell | N_1 : prime} (1 - \chi_D(\ell)/\ell) \prod_{\ell | N_2 : prime} (1 - \chi_D(\ell)) \cdot L(0, \chi_D)^2 \pmod 3.$$

Since the analytic class number formula shows that $L(0, \chi_D)$ equals the class number $h(D)$ of $\mathbb{Q}(\sqrt{D})$, up to a 3-adic unit, the indivisibility of $h(D)$ by 3 implies that we have $L\left(1, f_E \otimes \chi_D\right) \neq 0$ for a fundamental discriminant D with $(D, Np) = 1$, $\chi_D(\ell) = -1$ for each prime $\ell \mid N_2$ and $\chi_D(\ell)/\ell \equiv -1 \pmod 3$ for each prime $\ell \mid N_1$ by the theorem above (see [30, Corollary 3.4]). Then, Vatsal showed that $M_{f_E}(X) \gg X$ (see [30, Corollary 3.5]) by using a theorem of Nakagawa and Horie [21] to estimate a proportion of fundamental discriminants D satisfying $3 \nmid h(D)$ and the conditions which we mentioned above. By Corollary 5.9, we have the following:

Theorem 6.4. *Let $N \geq 3$ be a square-free odd integer and E an elliptic curve over $\mathbb{Q}$ of conductor N. Assume that E has a rational potint of order 3, that E has good ordinary reduction at 3 and that if ℓ is a prime at which E has non-split multiplicative reduction, then $\ell \equiv 2 \pmod 3$. Let f_E^* be the 3-stabilization of f_E. Then there exists a positive integer r and for a non-negative integer k with $k \equiv 0 \pmod{2 \cdot 3^r}$, there exists a 3-ordinary primitive form $f_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N, \mathbb{1}; \mathbb{Q}_{f_E^*})_0$ such that for any embedding σ of $\mathbb{Q}_{f_E^*}$ into $\mathbb{C}$, we have $M_{f^\sigma_{2k+2}}(X) \gg X$, where f^*_{2k+2} lies in a Coleman family passing through f_E^* and $f^\sigma_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N, \mathbb{1})$ is the primitive form defined by $a_n(f^\sigma_{2k+2}) := a_n(f_{2k+2})^\sigma$.*

Proof. Let D be a negative fundamental discriminant with $(D, Np) = 1$, $\chi_D(\ell) = -1$ for each prime $\ell \mid N_2$ and $\chi_D(\ell)/\ell \equiv -1 \pmod 3$ for each prime $\ell \mid N_1$. By assumption, we have $\chi_D(\ell) = 1$ for each prime $\ell \mid N_1$. Recall that $a_\ell(f_E) = -1$ if $\ell \mid N_1$ and $a_\ell(f_E) = 1$ if $\ell \mid N_2$. We thus have $\chi_D(\ell) = -a_\ell(f_E) = w_\ell(f)$ for any prime $\ell \mid N$ (see (27)), and hence $R_D(f_E) \neq 0$. Then there exists a primitive form $f_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N, \mathbb{1}; \mathbb{Q}_{f_E^*})_0$ satisfying $M_{f_{2k+2}}(X) \gg X$ by Corollary 5.9. For any isomorphism σ of $\mathbb{Q}_{f_E^*}$ into $\mathbb{C}$, we see that $f^\sigma_{2k+2} \in S^{\mathrm{new}}_{2k+2}(N, \mathbb{1})$ is a primitive form by [26, Proposition 1.2] and the theorem holds by [28, Theorem 1].

Example 6.5. Let E be the elliptic curve over $\mathbb{Q}$ given by the equation $y^2 + y = x^3 + x^2 - 9x - 15$. Then E has a rational point of order 3 and good ordinary reduction at 3 and is of conductor 19 ([30, Example 3.7]).

Moreover, E has split multiplicative reduction at 19, and hence E satisfies the assumption of the theorem above. Furthermore, equations

$$y^2 + y = x^3 + x^2 + 9x + 1, \tag{172}$$

$$y^2 + y = x^3 + x^2 - 23x - 50, \tag{173}$$

$$y^2 + y = x^3 + x^2 - x - 1, \tag{174}$$

$$y^2 + y = x^3 + x^2 - 49x + 600, \tag{175}$$

give elliptic curves over $\mathbb{Q}$ of conductor 35, 37, 51, and 77, respectively. They have split multiplicative reduction at any prime factor of their conductor and satisfy the assumption of the theorem above.

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