On analytic properties of L-functions attached to cusp forms on the unitary group of degree two

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Abstract

Let f be a holomorphic cusp form on $U(1,1)$. In this paper, we study the standard L-function of f and show its functional equation under certain conditions.

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1 Introduction

Let f be a holomorphic cusp form on $U(1,1)$. In this paper, we study the standard L-function of f and show its functional equation under certain conditions. The proof is based on the method of Shimura ([Sh2]).

We explain our results in a classical formulation in the simplest case. Let K be an imaginary quadratic field of discriminant D with $|D| > 4$. For simplicity, we assume that D is odd and the class number of K is 1. Let ℓ be a positive integer divisible by w_K, the number of roots of unity in K. Let

$$f(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}$$

be a holomorphic cusp form of weight $\ell - 1$ and character $\left(\dfrac{D}{*}\right)$ on $\Gamma_0(|D|)$. The L-function of f we are concerned with is given by

$$\begin{aligned} L(f;s) &= \zeta(2s) \sum_{\mathfrak{a}} c(\mathrm{N}\,\mathfrak{a}) \alpha^{\ell}\, \mathrm{N}\,\mathfrak{a}^{-(s+\ell-1)} \\ &\quad \times \prod_{p|D} (1-p^{-2s}) \left(1 - \overline{c(p)\Pi_p^{\ell}} p^{-s-\ell+1}\right)^{-1}, \end{aligned}$$

where $\mathfrak{a} = (\alpha)$ runs over the nonzero integral ideals of K and, for $p \mid D$, Π_p is a generator of the prime ideal of K dividing p.

Theorem *Suppose that f is a Hecke eigenform with $c(1) \neq 0$ and an eigenform of Atkin-Lehner operator at each $p \mid D$. Set*

$$L^*(f;s) = (2\pi)^{-2s} |D|^s \Gamma(s+1)\Gamma(s+\ell-1)L(f;s),$$

where $\Gamma(s)$ is the gamma function. Then $L^*(f;s)$ is continued to an entire function of s on $\mathbf{C}$ and satisfies

$$L^*(f;s) = L^*(f;1-s).$$

The result in a general case is stated in Section 4 in an adelic formulation. The main ingredients of the proofs are the Rankin-Selberg convolution and the local theory of Whittaker functions.

The paper is organized as follows. In Section 2, we prepare some notations used throughout this paper. In Section 3, we recall the definitions of cusp forms, Hecke operators, Atkin-Lehner operators, automorphic L-functions, theta series and Eisenstein series on $U(1,1)$. We also calculate the Fourier coefficients of the Eisenstein series. It is to be noted that a similar calculation was made by Shimura ([Sh2]) in a much more general situation. The main results of this paper are given in Section 4. In Section 5, we study the local and global Whittaker functions. Using the Rankin-Selberg convolution together with the results in Section 5, we give proofs of our results in Section 6. We state the classical interpretations of cusp forms in Section 7. Finally, in Section 8, we present an example.

2 Preliminaries

2.1 Notations

As usual, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ denote the ring of rational integers, the rational number field, the real number field and the complex number field respectively. We write i for $\sqrt{-1}$. For a ring R, $R^\times$ denotes the group of all invertible elements of R. Let $\mathbf{Z}_+$ and $\mathbf{R}_+$ denote the set of positive rational integers and that of positive real numbers respectively. For a set S, char_S stands for the characteristic function of S. For a prime v of $\mathbf{Q}$, $\mathbf{Q}_v$ denotes the completion of $\mathbf{Q}$ at v. For a finite prime p, $\mathbf{Z}_p$ denotes the p-adic integer ring. We put $\mathbf{Z}_f = \prod_{p<\infty} \mathbf{Z}_p$. Let $\mathbf{Q}_\mathbf{A}$ denote the adele ring of $\mathbf{Q}$. For $x \in \mathbf{Q}_\mathbf{A}$, let x_v be the v-component of x for each prime v. For a prime v, $|\cdot|_v$ stands for the absolute value of $\mathbf{Q}_v^\times$. For $v = \infty$, we often write $|\cdot|$ for $|\cdot|_\infty$. For $x=(x_v)_v \in \mathbf{Q}_\mathbf{A}^\times$, let $|x|_\mathbf{A} = \prod_{v\le\infty} |x_v|_v$ be the idele norm of x. For a finite prime p, we normalize the additive valuation $\mathrm{ord}_p : \mathbf{Q}_p^\times \to \mathbf{Z}$ so that $\mathrm{ord}_p\, p = 1$. In this paper, we fix an imaginary quadratic field K of discriminant D with integer ring $\mathcal{O}_K$. We only consider the case of $|D|>4$. Then we have $w_K=2$, where w_K is the number of roots of unity in K. Denote by σ the nontrivial automorphism of $K/\mathbf{Q}$. For $z\in K$, let $\mathrm{Tr}(z)=z+z^\sigma$ and $\mathrm{N}(z)=zz^\sigma$. The complex conjugate of $z\in\mathbf{C}$ is denoted by $\overline{z}$. For a prime v of $\mathbf{Q}$, let $K_v = K\otimes_\mathbf{Q} \mathbf{Q}_v$. For a finite prime p, we put $\mathcal{O}_{K,p} = \mathcal{O}_K \otimes_\mathbf{Z} \mathbf{Z}_p$. We set $\mathcal{O}_{K,f} = \prod_{p<\infty} \mathcal{O}_{K,p}$. We denote by $K_\mathbf{A}$ and $K_{\mathbf{A},f}$ the adele ring of K and its finite part respectively. For $a=(a_v)_v \in K_\mathbf{A}^\times$, put $\|a\|_\mathbf{A} = \prod_{v\le\infty} \|a_v\|_v$, where $\|a_v\|_v = |\mathrm{N}(a_v)|_v$. We put $K^1=\{t\in K^\times;\, \mathrm{N}(t)=1\}$ and $\mathcal{O}_{K,f}^1 = K_{\mathbf{A},f}^1 \cap \mathcal{O}_{K,f}^\times = \prod_{p<\infty} \mathcal{O}_{K,p}^1$, where $\mathcal{O}_{K,p}^1 = K_p^1 \cap \mathcal{O}_{K,p}^\times$. When p ramifies in $K/\mathbf{Q}$ (namely $p\mid D$), we fix a prime element Π_p of K_p. When p splits in $K/\mathbf{Q}$, we fix an identification between K_p and $\mathbf{Q}_p\oplus\mathbf{Q}_p$ and put $\Pi_{p,1}=(p,1)$ and $\Pi_{p,2}=(1,p)$.

2.2 Characters

Let $\mathcal{X}$ be the set of Hecke characters χ of K satisfying $\chi|_{\mathbf{Q}_\mathbf{A}^\times}=\omega$, where ω denotes the quadratic Hecke character of $\mathbf{Q}$ corresponding to $K/\mathbf{Q}$ by class field theory. For $\chi\in\mathcal{X}$, let $w_\infty(\chi)$ be the integer such that $\chi(z_\infty) = (z_\infty/|z_\infty|)^{w_\infty(\chi)}$ for $z_\infty\in\mathbf{C}^\times$. We fix an element $\chi_0\in\mathcal{X}$ such that $w_\infty(\chi_0)=-1$. Let ψ_v be the additive character of $\mathbf{Q}_v$ given by

$$\psi_v(x) = \begin{cases} e^{2\pi i x} & (v=\infty), \\ e^{-2\pi i \{x\}_p} & (v=p), \end{cases}$$

where $\{x\}_p$ denotes the fractional part of $x\in\mathbf{Q}_p$. Then $\psi = \prod_{v\le\infty}\psi_v$ is a nontrivial character of $\mathbf{Q}\backslash\mathbf{Q}_\mathbf{A}$. We put $\psi_{K_v} = \psi_v\circ\mathrm{Tr}$.

2.3 Unitary group

Let $H=U(T)$ be the unitary group of $T=\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Namely

$$H_\mathbf{Q} = \left\{ h\in \mathrm{GL}_2(K);\, {}^t h^\sigma T h = T \right\}.$$

We put

$$\boldsymbol{n}(x)=\begin{pmatrix}1 & x\\ & 1\end{pmatrix},\quad \boldsymbol{d}(y)=\begin{pmatrix}y^{\sigma} & \\ & y^{-1}\end{pmatrix},\quad \overline{\boldsymbol{n}}(z)=\begin{pmatrix}1 & \\ z & 1\end{pmatrix}$$
$$\text{and } S=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\quad (x\in\mathbf{Q},\ y\in K^{\times},\ z\in\mathbf{Q}).$$

Define subgroups of H by

$$N=\{\boldsymbol{n}(x);\, x\in\mathbf{Q}\},\quad Z=\{a\cdot I;\, a\in K^{1}\},$$
$$P=\{\boldsymbol{n}(x)\boldsymbol{d}(y);\, x\in\mathbf{Q},\, y\in K^{\times}\},$$

where I is the identity matrix of degree 2. We denote by $H_{\mathbf{Q}}$ (resp. H_v) the group of the $\mathbf{Q}$-rational (resp. $\mathbf{Q}_v$-rational) points of H. Let $H_{\mathbf{A}}$ be the adele group of H and $H_{\mathbf{A},f}$ the finite part of $H_{\mathbf{A}}$. For a finite prime p of $\mathbf{Q}$, let $\mathcal{U}_p=H_p\cap \mathrm{GL}_2(\mathcal{O}_{K,p})$ and $\mathcal{U}_0(D)_p=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathcal{U}_p;\, c\in D\mathcal{O}_{K,p}\right\}$. Note that $\mathcal{U}_0(D)_p=\mathcal{U}_p$ unless $p\mid D$. We put $\mathcal{U}_f=\prod_{p<\infty}\mathcal{U}_p$ and $\mathcal{U}_0(D)_f=\prod_{p<\infty}\mathcal{U}_0(D)_p$. Let $\mathcal{U}_\infty=\{h\in H_\infty;\, h\langle i\rangle=i\}$, where $h\langle z\rangle=\dfrac{az+b}{cz+d}$ for $h=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in H_\infty$ and $z\in\mathfrak{H}=\{z\in\mathbf{C};\,\mathrm{Im}(z)>0\}$. For $h\in H_\infty$ and $z\in\mathfrak{H}$, we define the automorphic factors by $j(h,z)=cz+d$ and $J(h,z)=(\det h)^{-1}j(h,z)$. Let $\widetilde{\chi_{0,p}}$ be the character of $\mathcal{U}_0(D)_p$ given by

$$\widetilde{\chi_{0,p}}\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\begin{cases}\chi_{0,p}(a) & (c\in p\mathcal{O}_{K,p}),\\ \chi_{0,p}(c) & (c\in\mathcal{O}_{K,p}-p\mathcal{O}_{K,p}).\end{cases}$$

Then $\widetilde{\chi_0}=\prod_{p<\infty}\widetilde{\chi_{0,p}}$ defines a character of $\mathcal{U}_0(D)_f$.

2.4 Measures

Let dx_p be the Haar measure on $\mathbf{Q}_p$ normalized by $\int_{\mathbf{Z}_p}dx_p=1$. Let dx_∞ be the usual Lebesgue measure on $\mathbf{R}$. Then $dx=\prod_{v\le\infty}dx_v$ is the Haar measure on $\mathbf{Q}_{\mathbf{A}}$ with $\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}}dx=1$. Let $d^{\times}x_p$ be the Haar measure on $\mathbf{Q}_p^{\times}$ normalized by $\int_{\mathbf{Z}_p^{\times}}d^{\times}x_p=1$. Let $d^{\times}x_\infty=|x_\infty|^{-1}dx_\infty$. Then $d^{\times}x=\prod_{v\le\infty}d^{\times}x_v$ is the Haar measure on $\mathbf{Q}_{\mathbf{A}}^{\times}$. For each prime v of $\mathbf{Q}$, let dy_v be the Haar measure on K_v self-dual with respect to the pairing $(x,y)\mapsto\psi_{K_v}(xy^{\sigma})$. Note that $\int_{\mathcal{O}_{K,p}}dy_p=|D|_p^{1/2}$ if $v=p<\infty$, and that dy_∞ is twice the usual Lebesgue measure on $\mathbf{C}$. Then $dy = \prod_{v \leq \infty} dy_v$ is the Haar measure on $K_{\mathbf{A}}$ with $\int_{K \backslash K_{\mathbf{A}}} dy = 1$. Let $d^\times y = \prod_{v \leq \infty} d^\times y_v$ be the Haar measure on $K_{\mathbf{A}}^\times$, where $d^\times y_v$ is the Haar measure on $K_v^\times$ normalized by $\int_{\mathcal{O}_{K,p}^\times} d^\times y_p = 1$ if $v = p < \infty$, and $d^\times y_\infty = 2^{-1}\,\mathrm{N}(y_\infty)^{-1} dy_\infty$. For a prime v of $\mathbf{Q}$, we normalize the Haar measure dh_v on H_v by

$$\int_{H_v} f(h_v) dh_v = \int_{\mathbf{Q}_v} dx_v \int_{K_v^\times} d^\times y_v \int_{\mathcal{U}_v} du_v \; \|y_v\|_v^{-1} f(\boldsymbol{n}(x_v)\boldsymbol{d}(y_v)u_v),$$

where du_p and du_∞ are the Haar measures on $\mathcal{U}_p$ and $\mathcal{U}_\infty$ normalized by $\int_{\mathcal{U}_0(D)_p} du_p = \int_{\mathcal{U}_\infty} du_\infty = 1$, respectively. Then $dh = \prod_{v \leq \infty} dh_v$ is the Haar measure on $H_{\mathbf{A}}$.

3 Definitions

3.1 Cusp forms

Let $\ell \in \mathbf{Z}_+$ with $w_K \mid \ell$. A smooth function f on $H_{\mathbf{Q}} \backslash H_{\mathbf{A}}$ is called a *cusp form on* $\mathcal{U}_0(D)_f$ *of weight* $\ell - 1$ *with character* χ_0 if the following conditions (1)–(4) are satisfied.

(1) $f(hu_f u_\infty) = \widetilde{\chi_0}(u_f) J(u_\infty, i)^{1-\ell} f(h)$ $(h \in H_{\mathbf{A}},\ u_f \in \mathcal{U}_0(D)_f,\ u_\infty \in \mathcal{U}_\infty)$.

(2) For every $h_f \in H_{\mathbf{A},f}$, the function $f_{\mathrm{dm},h_f} : \mathfrak{H} \ni h_\infty \langle i \rangle \mapsto J(h_\infty, i)^{\ell-1} f(h_\infty h_f)$ $(h_\infty \in H_\infty)$ is holomorphic.

(3) For every $h_f \in H_{\mathbf{A},f}$, f_{dm,h_f} is holomorphic at $i\infty$.

(4) $\int_{\mathbf{Q} \backslash \mathbf{Q}_{\mathbf{A}}} f(\boldsymbol{n}(x)h) dx = 0$ $(h \in H_{\mathbf{A}})$.

We denote by $S_{\ell-1}(D, \chi_0)$ the space of such functions. Let $\mathcal{Y}_\ell$ be the set of characters Ω of $K_{\mathbf{A}}^1 / K^1$ satisfying $\Omega|_{\mathcal{O}_{K,f}^1} = \mathbf{1}$ and $\Omega(z_\infty) = z_\infty^\ell$ for $z_\infty \in \mathbf{C}^1$. For $\Omega \in \mathcal{Y}_\ell$, we put

$$S_{\ell-1}(D, \chi_0; \chi_0\Omega) = \left\{ f \in S_{\ell-1}(D, \chi_0);\ f(th) = (\chi_0\Omega)(t) f(h)\ (t \in K_{\mathbf{A}}^1,\ h \in H_{\mathbf{A}}) \right\}.$$

Then $S_{\ell-1}(D, \chi_0) = \bigoplus_{\Omega \in \mathcal{Y}_\ell} S_{\ell-1}(D, \chi_0; \chi_0\Omega)$.

For $f \in S_{\ell-1}(D, \chi_0; \chi_0\Omega)$, we define the *global Whittaker function* W_f attached to f by

$$W_f(h) = \int_{\mathbf{Q} \backslash \mathbf{Q}_{\mathbf{A}}} \psi(-x) f(\boldsymbol{n}(x)h) dx \quad (h \in H_{\mathbf{A}}).$$

It is easily seen that

$$W_f(t\boldsymbol{n}(x)hu_fu_\infty) = (\chi_0\Omega)(t)\psi(x)\widetilde{\chi_0}(u_f)J(u_\infty, i)^{1-\ell}W_f(h)$$

for $t \in K^1_{\mathbf{A}}$, $x \in \mathbf{Q}_{\mathbf{A}}$, $u_f \in \mathcal{U}_0(D)_f$ and $u_\infty \in \mathcal{U}_\infty$.

3.2 Hecke operators and Atkin-Lehner operators

Let $f \in S_{\ell-1}(D, \chi_0; \chi_0\Omega)$. For each finite prime p, we define *Hecke operators* as follows.

(i) Suppose that p is inert in $K/\mathbf{Q}$. Then we put

$$\mathcal{T}_p f(h) = -f(h\boldsymbol{d}(p^{-1})) - \sum_{x \in \mathbf{Z}_p^\times/p\mathbf{Z}_p} f(h\boldsymbol{n}(p^{-1}x)) - \sum_{y \in \mathbf{Z}_p/p^2\mathbf{Z}_p} f(h\boldsymbol{n}(y)\boldsymbol{d}(p)).$$

(ii) Suppose that p ramifies in $K/\mathbf{Q}$. Then we put

$$\mathcal{T}_p f(h) = \chi_{0,p}(\Pi_p)^{-1} \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1})) + \chi_{0,p}(\Pi_p) \sum_{y \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p)).$$

(iii) Suppose that p splits in $K/\mathbf{Q}$. Then we put

$$\begin{aligned} \mathcal{T}_{p,1} f(h) &= \chi_{0,p}(\Pi_{p,1})^{-1} \left\{ f(h\boldsymbol{d}(\Pi_{p,1}^{-1})) + \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\boldsymbol{n}(x)\boldsymbol{d}(\Pi_{p,2})) \right\}, \\ \mathcal{T}_{p,2} f(h) &= \chi_{0,p}(\Pi_{p,2})^{-1} \left\{ f(h\boldsymbol{d}(\Pi_{p,2}^{-1})) + \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\boldsymbol{n}(x)\boldsymbol{d}(\Pi_{p,1})) \right\}. \end{aligned}$$

Remark 3.1 Note that $S_{\ell-1}(D, \chi_0; \chi_0\Omega)$ is invariant under Hecke operators, and $\mathcal{T}_{p,2}f = \Omega(\Pi_{p,2}/\Pi_{p,1})\mathcal{T}_{p,1}f$ for p split in $K/\mathbf{Q}$.

We say that $f \in S_{\ell-1}(D, \chi_0; \chi_0\Omega)$ is a *Hecke eigenform* with eigenvalues $\{\Lambda_p\}$ ($\Lambda_p = (\Lambda_{p,1}, \Lambda_{p,2}) \in \mathbf{C}^2$ if p splits in $K/\mathbf{Q}$ and $\Lambda_p \in \mathbf{C}$ if p does not split in $K/\mathbf{Q}$) if, for every $p < \infty$, $\mathcal{T}_p f = \Lambda_p f$ in the non-split case and $\mathcal{T}_{p,j} f = \Lambda_{p,j} f$ ($j = 1, 2$) in the split case. Note that $\Lambda_{p,2} = \Omega(\Pi_{p,2}/\Pi_{p,1})\Lambda_{p,1}$ in the split case.

For each prime factor p of D, we put $w_{D,p} = \begin{pmatrix} & \sqrt{D}^{-1} \\ \sqrt{D} & \end{pmatrix} \in H_p$. We define the *Atkin-Lehner operator* $\mathfrak{F}_{D,p}$ by

$$(\mathfrak{F}_{D,p} f)(h) = f(hw_{D,p})$$

for $f \in S_{\ell-1}(D, \chi_0; \chi_0\Omega)$ (cf. [A-L]). Then $f \mapsto \mathfrak{F}_{D,p} f$ defines an involution of $S_{\ell-1}(D, \chi_0; \chi_0\Omega)$ commuting with Hecke operators and the eigenvalues of $\mathfrak{F}_{D,p}$ are $\{\pm 1\}$.

3.3 L-function

Let $f \in S_{\ell-1}(D, \chi_0; \chi_0\Omega)$ be a Hecke eigenform with eigenvalues $\{\Lambda_p\}$. Let k be a positive integer with $w_K \mid k$ and $k \geq \ell$, and let Ξ a Hecke character of K satisfying

$$\Xi|_{\mathcal{O}_{K,f}^{\times}} = \mathbf{1} \tag{3.1}$$

and

$$\Xi(z_\infty) = (z_\infty / |z_\infty|)^{k-\ell} \tag{3.2}$$

for $z_\infty \in \mathbf{C}^{\times}$. Here $\mathbf{1}$ is the trivial character. We define the automorphic L-function $L(f, \Xi; s)$ by

$$L(f, \Xi; s) = \prod_{p<\infty} L_p(f, \Xi_p; s)$$

with $s \in \mathbf{C}$. Here the local factor $L_p(f, \Xi_p; s)$ is given as follows:

$$L_p(f, \Xi_p; s) = \begin{cases} \left(1 + (1 - p - \Lambda_p)\Xi_p(p)p^{-2s-1} + \Xi_p(p)^2 p^{-4s}\right)^{-1} & (p \text{ is inert in } K/\mathbf{Q}), \\ \prod_{j=1,2} \left(1 - \Lambda_{p,j}\Xi_p(\Pi_{p,j})p^{-s-1/2} + \Omega(\Pi_{p,j}/\Pi_{p,j}^{\sigma})\Xi_p(\Pi_{p,j})^2 p^{-2s}\right)^{-1} & (p \text{ splits in } K/\mathbf{Q}), \\ \left(1 - \Lambda_p \Xi_p(\Pi_p)p^{-s-1/2} + \Xi_p(\Pi_p)^2 p^{-2s}\right)^{-1} & (p \text{ ramifies in } K/\mathbf{Q}). \end{cases}$$

3.4 Metaplectic representation and theta series

Let φ_v be a Schwartz-Bruhat function on K_v. Let

$$\widehat{\varphi_v}(z) = \int_{K_v} \psi_{K_v}(y_v^{\sigma} z)\varphi_v(y_v)dy_v$$

be the Fourier transform of φ_v. We define the Weil constant $\lambda_{K,v}(\psi_v)$ by

$$\int_{K_v} \varphi_v(z_v)\psi_v(\mathrm{N}(z_v))dz_v = \lambda_{K,v}(\psi_v)\int_{K_v} \widehat{\varphi_v}(z_v)\psi_v(-\mathrm{N}(z_v))dz_v$$

(cf. [We2]). The following facts are well-known.

Lemma 3.2

(1) $\lambda_{K,v}(\psi_v)^2 = \omega_v(-1)$ *for every* v.

(2) *For a finite prime p of $\mathbf{Q}$, we have*

$$\lambda_{K,p}(\psi_p) = \begin{cases} 1 & (p \nmid D), \\ \sqrt{p}^{\operatorname{ord}_p D} \displaystyle\int_{\mathbf{Z}_p^\times} \omega_p(p^{-\operatorname{ord}_p D} t)\psi_p(p^{-\operatorname{ord}_p D} t)dt & (p \mid D). \end{cases}$$

(3) $\lambda_{K,\infty}(\psi_\infty) = i.$

(4) $\displaystyle\prod_{v \le \infty} \lambda_{K,v}(\psi_v) = 1.$

Let $\mathcal{S}(K_{\mathbf{A}})$ be the space of Schwartz-Bruhat functions on $K_{\mathbf{A}}$. Let χ_1 be an element of $\mathcal{X}$ such that $w_\infty(\chi_1) = 2k+1$ and $\chi_0^{-1}\chi_1|_{\mathcal{O}_{K,f}^\times} = \mathbf{1}$. It is known that there exists a smooth representation $\mathcal{M}_{\chi_1}^T$ of $H_{\mathbf{A}}$ on $\mathcal{S}(K_{\mathbf{A}})$ determined by

$$\begin{aligned} \mathcal{M}_{\chi_1}^T(\boldsymbol{d}(a))\varphi(X) &= \chi_1(a)^{-1}\|a\|_{\mathbf{A}}^{1/2}\varphi(aX) \quad (a \in K_{\mathbf{A}}^\times), \\ \mathcal{M}_{\chi_1}^T(\boldsymbol{n}(b))\varphi(X) &= \psi(b\,\mathrm{N}(X))\varphi(X) \quad (b \in \mathbf{Q}_{\mathbf{A}}), \\ \mathcal{M}_{\chi_1}^T(S_v)\varphi(X) &= \lambda_{K,v}(\psi_v)\int_{K_v} \psi_{K_v}(Y_v^\sigma X_v)\varphi(Y_v X^{(v)})dY_v \quad (v \le \infty). \end{aligned}$$

Here $S_v \in H_v$ and $X^{(v)} = \displaystyle\prod_{v' \neq v} X_{v'}$. We call $\mathcal{M}_{\chi_1}^T$ the *metaplectic representation* of $H_{\mathbf{A}}$. Let $\varphi_0 = \displaystyle\bigotimes_{v \le \infty} \varphi_{0,v} \in \mathcal{S}(K_{\mathbf{A}})$, where

$$\begin{aligned} \varphi_{0,p}(X_p) &= \mathrm{char}_{\mathcal{O}_{K,p}}(X_p), \\ \varphi_{0,\infty}(X_\infty) &= X_\infty^k \exp(-2\pi |X_\infty|^2). \end{aligned}$$

It is known that

$$\mathcal{M}_{\chi_1}^T(u_v)\varphi_{0,v} = \begin{cases} \widetilde{\chi_{0,p}}(u_p)\varphi_{0,p} & (v = p < \infty,\ u_p \in \mathcal{U}_0(D)_p), \\ J(u_\infty, i)^{-k-1}\varphi_{0,\infty} & (v = \infty,\ u_\infty \in \mathcal{U}_\infty). \end{cases}$$

We define a *theta series* by

$$\theta_{\chi_1}(h) = \sum_{X \in K} \mathcal{M}_{\chi_1}^T(h)\varphi_0(X).$$

Then $\theta_{\chi_1} \in S_{k+1}(D, \chi_0)$.

3.5 Eisenstein series

In what follows, we fix an $\Omega \in \mathcal{Y}_\ell$. Let ξ be the Hecke character of K given by

$$\xi(z) = (\chi_0\chi_1^{-1}\Xi)(z)\Omega(z/z^\sigma) \quad (z \in K_{\mathbf{A}}^\times). \tag{3.3}$$

Note that $\xi|_{\mathcal{O}_{K,f}^\times} = \mathbf{1}$ and $\xi(z_\infty) = (z_\infty/|z_\infty|)^{-k+\ell-2}$ for $z_\infty \in \mathbf{C}^\times$. For $s \in \mathbf{C}$, we define an *Eisenstein series* by

$$E_{k-\ell+2}(h, \Xi; s) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash H_{\mathbf{Q}}} \phi_{k-\ell+2}(\gamma h; s) \qquad (k \geq \ell),$$

where

$$\phi_{k-\ell+2}\left(\boldsymbol{n}(b)\boldsymbol{d}(a)u_f u_\infty; s\right) = \xi(a)\left\|a\right\|_{\mathbf{A}}^s J(u_\infty, i)^{-k+\ell-2}$$

for $a \in K_{\mathbf{A}}^\times$, $b \in \mathbf{Q}_{\mathbf{A}}$, $u_f \in \mathcal{U}_f$ and $u_\infty \in \mathcal{U}_\infty$. The series $E_{k-\ell+2}(h, \Xi; s)$ converges absolutely and uniformly for $(h, s) \in C \times C'$, where C (resp. C') is any compact subset of $H_{\mathbf{A}}$ (resp. of $\{s \in \mathbf{C}; \mathrm{Re}(s) > 1\}$). Note that

$$E_{k-\ell+2}(\gamma t h u_f u_\infty, \Xi; s) = \xi(t^\sigma) J(u_\infty, i)^{-k+\ell-2} E_{k-\ell+2}(h, \Xi; s)$$

for $\gamma \in H_{\mathbf{Q}}$, $t \in K_{\mathbf{A}}^1$, $u_f \in \mathcal{U}_f$ and $u_\infty \in \mathcal{U}_\infty$.

Proposition 3.3 *Let* $P_r(s) = \prod_{j=0}^{r}(s + r - j)$. *Put*

$$E^*_{k-\ell+2}(h, \Xi; s) = \pi^{-s}\Gamma(s)\zeta(2s)P_{(k-\ell)/2}(s)E_{k-\ell+2}(h, \Xi; s).$$

Then $E^*_{k-\ell+2}(h, \Xi; s)$ *is continued to an entire function of* s *on* $\mathbf{C}$*, and satisfies a functional equation*

$$E^*_{k-\ell+2}(h, \Xi; s) = E^*_{k-\ell+2}(h, \Xi; 1 - s).$$

Proof. We first prepare for the proof of this proposition. For simplicity, we write $\kappa = k - \ell$. Put $G(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. The function $G(s)$ is holomorphic on $\mathbf{C}$ except for $s = 0$ and $s = 1$, and has a functional equation $G(s) = G(1 - s)$.

By the definition of ξ, we have the following. Since $\Pi_p^2 \in p\mathcal{O}_{K,p}^\times$ for $p \mid D$, we have $\xi_p(\Pi_p)^2 = \xi_p(p)$. The equation $1 = \xi(p) = \xi_\infty(p)\xi_p(p) = \xi_p(p)$ implies that $\xi_p(p) = 1$ for all p. Moreover we have $\xi(a^\sigma)^{-1} = \xi(a)$ for $a \in K_{\mathbf{A}}^\times$ since $\xi(\mathrm{N}(a)) = \Xi(\mathrm{N}(a))$ and $\mathrm{N}(a) \in \mathbf{Q}_{\mathbf{A}}^\times = \mathbf{Q}^\times\mathbf{R}_+ \prod_{p<\infty} \mathbf{Z}_p^\times$.

We define the classical Whittaker function $W_{\nu,\mu}(z)$ by

$$W_{\nu,\mu}(z)=\frac{e^{-z/2}z^{\mu+1/2}}{\Gamma(\mu-\nu+1/2)}\int_0^{\infty}e^{-zt}t^{\mu-\nu-1/2}(1+t)^{\mu+\nu-1/2}dt$$

for ν, μ ($\mathrm{Re}(\mu-\nu+1/2)>0$) and $z\in\mathbf{C}$ ($|\arg(z)|<\pi$). It is well-known that $W_{\nu,\mu}(z)$ is continued to an entire function of (ν,μ) on $\mathbf{C}^2$, and satisfies a functional equation $W_{\nu,\mu}(z)=W_{\nu,-\mu}(z)$.

We now calculate the Fourier coefficients of $E_{\kappa+2}(h,\Xi;s)$:

$$E_{\kappa+2}(h,\Xi;s)=\sum_{m\in\mathbf{Q}}E^{(m)}_{\kappa+2}(h,\Xi;s),$$

where

$$E^{(m)}_{\kappa+2}(h,\Xi;s)=\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}}\psi(-mx)E_{\kappa+2}(\boldsymbol{n}(x)h,\Xi;s)dx.$$

From now on, we fix an $h=\boldsymbol{n}(b)\boldsymbol{d}(a)u_fu_\infty\in H_{\mathbf{A}}$ ($b\in\mathbf{Q}_{\mathbf{A}}$, $a\in K^{\times}_{\mathbf{A}}$, $u_f\in\mathcal{U}_f$, $u_\infty\in\mathcal{U}_\infty$). By the Bruhat decomposition $H_{\mathbf{Q}}=P_{\mathbf{Q}}\cup P_{\mathbf{Q}}SN_{\mathbf{Q}}$, we have

$$\begin{aligned}E^{(m)}_{\kappa+2}(h,\Xi;s) &= \int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}}\psi(-mx)\left\{\phi_{\kappa+2}(\boldsymbol{n}(x)h;s)+\sum_{r\in\mathbf{Q}}\phi_{\kappa+2}(S\boldsymbol{n}(x+r)h;s)\right\}dx\\ &= \phi_{\kappa+2}(h;s)\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}}\psi(-mx)dx\\ &\quad+\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}}\sum_{r\in\mathbf{Q}}\psi(mr)\psi(-m(x+r))\phi_{\kappa+2}(S\boldsymbol{n}(x+r)h;s)dx\\ &= \delta_{m,0}\phi_{\kappa+2}(h;s)+I^{(m)}(h;s),\end{aligned}$$

where $\delta_{a,b}$ is the Kronecker's delta and

$$I^{(m)}(h;s)=\int_{\mathbf{Q}_{\mathbf{A}}}\psi(-mx)\phi_{\kappa+2}(S\boldsymbol{n}(x)h;s)dx.$$

Since

$$I^{(m)}(\boldsymbol{n}(b)\boldsymbol{d}(a)u_fu_\infty;s)=\psi(mb)J(u_\infty,i)^{-\kappa-2}I^{(m)}(\boldsymbol{d}(a);s)$$

for $b\in\mathbf{Q}_{\mathbf{A}}$, $a\in K^{\times}_{\mathbf{A}}$, $u_f\in\mathcal{U}_f$ and $u_\infty\in\mathcal{U}_\infty$, we only have to consider $I^{(m)}(\boldsymbol{d}(a);s)$ ($a\in K^{\times}_{\mathbf{A}}$). Decompose $I^{(m)}(\boldsymbol{d}(a);s)$ as $I^{(m)}(\boldsymbol{d}(a);s)=\prod_{v\leq\infty}I^{(m)}_v(\boldsymbol{d}(a_v);s)$,

$$I^{(m)}_v(\boldsymbol{d}(a_v);s)=\int_{\mathbf{Q}_v}\psi_v(-mx)\phi_{\kappa+2}(S_v\boldsymbol{n}(x)_v\boldsymbol{d}(a_v);s)dx.$$

For $m \in \mathbf{Q}$ and $M \in \mathbf{Z}$, put $\mu = \operatorname{ord}_p m$ and $R_p(m, M; s) = \sum_{n=0}^{M+\mu} p^{(-2s+1)n}$ at a finite prime p.

Suppose that p is inert. For $a_p \in K_p^\times$, put $\operatorname{ord}_K a_p = A$ if $a_p \in p^A \mathcal{O}_{K,p}^\times$. It is easily seen that

$$\begin{aligned} I_p^{(m)}(\boldsymbol{d}(a_p); s) &= \int_{p^{2A}\mathbf{Z}_p} \psi_p(-mx)\xi_p(p^{-A}) \left\| p^{-A} \right\|_p^s dx \\ &\quad + \int_{\mathbf{Q}_p - p^{2A}\mathbf{Z}_p} \psi_p(-mx)\xi_p(p^{A-\operatorname{ord}_p x}) \left\| p^{A-\operatorname{ord}_p x} \right\|_p^s dx \\ &= \xi_p(p)^{-A} p^{2As} \int_{p^{2A}\mathbf{Z}_p} \psi_p(-mx) dx \\ &\quad + \xi_p(p)^A p^{-2As} \sum_{n=-\infty}^{2A-1} \xi_p(p)^{-n} p^{(2s-1)n} \int_{\mathbf{Z}_p^\times} \psi_p(-mp^n x) dx. \end{aligned}$$

If $m = 0$, we have

$$\begin{aligned} & I_p^{(0)}(\boldsymbol{d}(a_p); s) \\ &= \xi_p(p)^{-A} p^{2A(s-1)} + \xi_p(p)^A (1 - p^{-1}) p^{-2As} \sum_{n=-\infty}^{2A-1} \xi_p(p)^{-n} p^{(2s-1)n} \\ &= \xi_p(p)^{-A} p^{2A(s-1)} \left\{ 1 + \xi_p(p)^{2A} (1 - p^{-1}) p^{-2A(2s-1)} \sum_{n=-2A+1}^{\infty} \xi_p(p)^n p^{(-2s+1)n} \right\} \\ &= \xi_p(p)^{-A} p^{2A(s-1)} \left\{ 1 + \xi_p(p)(1 - p^{-1}) p^{-2s+1} \sum_{n=0}^{\infty} \xi_p(p)^n p^{(-2s+1)n} \right\} \\ &= \xi_p(p)^{-A} p^{2A(s-1)} \left\{ 1 + \xi_p(p)(1 - p^{-1}) p^{-2s+1} \left(1 - \xi_p(p) p^{-2s+1}\right)^{-1} \right\} \\ &= \xi_p(p)^{-A} p^{2A(s-1)} \left(1 - \xi_p(p) p^{-2s+1}\right)^{-1} \left(1 - \xi_p(p) p^{-2s}\right). \end{aligned}$$

If $m \notin p^{-2A}\mathbf{Z}_p$, we obtain

$$I_p^{(m)}(\boldsymbol{d}(a_p); s) = 0.$$

If $m \in p^{-2A}\mathbf{Z}_p$, we have

$$
\begin{aligned}
& I_p^{(m)}(\boldsymbol{d}(a_p); s) \\
&= \xi_p(p)^{-A} p^{2A(s-1)} \\
&\quad - \xi_p(p)^{A+\mu+1} p^{-2As+(-2s+1)(\mu+1)} \left\{ 1 - (1-p^{-1}) \sum_{n=0}^{2A+\mu} \xi_p(p)^{-n} p^{(2s-1)n} \right\} \\
&= \xi_p(p)^{-A} p^{2A(s-1)} \Bigg[1 - \xi_p(p)^{2A+\mu+1} p^{(-2s+1)(2A+\mu+1)} \\
&\quad \times \left\{ 1 - (1-p^{-1}) \frac{1 - \xi_p(p)^{-2A-\mu-1} p^{(2s-1)(2A+\mu+1)}}{1 - \xi_p(p)^{-1} p^{2s-1}} \right\} \Bigg] \\
&= \xi_p(p)^{-A} p^{2A(s-1)-1} \left(1 - \xi_p(p)^{-1} p^{2s-1}\right)^{-1} \\
&\quad \times \left(1 - \xi_p(p)^{-1} p^{2s}\right) \left(1 - \xi_p(p)^{2A+\mu+1} p^{(-2s+1)(2A+\mu+1)}\right) \\
&= \xi_p(p)^{-A} p^{2A(s-1)} \left(1 - \xi_p(p) p^{-2s+1}\right)^{-1} \\
&\quad \times \left(1 - \xi_p(p) p^{-2s}\right) \left(1 - \xi_p(p)^{2A+\mu+1} p^{(-2s+1)(2A+\mu+1)}\right) \\
&= \xi_p(p)^{-A} p^{2A(s-1)} \left(1 - \xi_p(p) p^{-2s}\right) R_p(m, 2A; s).
\end{aligned}
$$

Hence we see that

$$
I_p^{(m)}(\boldsymbol{d}(a_p); s) = \begin{cases} p^{2A(s-1)} \left(1 - p^{-2s+1}\right)^{-1} \left(1 - p^{-2s}\right) & (m = 0), \\ p^{2A(s-1)} \left(1 - p^{-2s}\right) R_p(m, 2A; s) & (m \in p^{-2A}\mathbf{Z}_p), \\ 0 & (m \notin p^{-2A}\mathbf{Z}_p), \end{cases}
$$

where $A = \mathrm{ord}_K a_p$ and $\mu = \mathrm{ord}_p m$.

Suppose that p ramifies. For $a_p \in K_p^\times$, put $A = \mathrm{ord}_K a_p$ if $a_p \in \Pi_p^A \mathcal{O}_{K,p}^\times$. It is easily seen that

$$
\begin{aligned}
I_p^{(m)}(\boldsymbol{d}(a_p); s) &= \int_{p^A\mathbf{Z}_p} \psi_p(-mx) \xi_p(\Pi_p^{-A}) \left\| \Pi_p^{-A} \right\|_p^s dx \\
&\quad + \int_{\mathbf{Q}_p - p^A\mathbf{Z}_p} \psi_p(-mx) \xi_p(\Pi_p^{A-2\,\mathrm{ord}_p x}) \left\| \Pi_p^{A-2\,\mathrm{ord}_p x} \right\|_p^s dx \\
&= \xi_p(\Pi_p)^{-A} p^{As} \int_{p^A\mathbf{Z}_p} \psi_p(-mx) dx \\
&\quad + \xi_p(\Pi_p)^A p^{-As} \sum_{n=-\infty}^{A-1} \xi_p(\Pi_p)^{-2n} p^{(2s-1)n} \int_{\mathbf{Z}_p^\times} \psi_p(-mp^n x) dx.
\end{aligned}
$$

If $m=0$, we have

$$\begin{aligned} & I_p^{(0)}(\boldsymbol{d}(a_p);s) \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)} + (1-p^{-1})\xi_p(\Pi_p)^{A}p^{-As}\sum_{n=-A+1}^{\infty}\xi_p(\Pi_p)^{2n}p^{(-2s+1)n} \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)}\left\{1+(1-p^{-1})\xi_p(\Pi_p)^{2}p^{-2s+1}\sum_{n=0}^{\infty}\xi_p(\Pi_p)^{2n}p^{(-2s+1)n}\right\} \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)}\left(1-\xi_p(\Pi_p)^{2}p^{-2s+1}\right)^{-1}\left(1-\xi_p(\Pi_p)^{2}p^{-2s}\right). \end{aligned}$$

If $m\notin p^{-A}\mathbf{Z}_p$, we obtain

$$I_p^{(m)}(\boldsymbol{d}(a_p);s)=0.$$

If $m\in p^{-A}\mathbf{Z}_p$, we have

$$\begin{aligned} & I_p^{(m)}(\boldsymbol{d}(a_p);s) \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)} \\ & -\xi_p(\Pi_p)^{A+2\mu+2}p^{-As+(-2s+1)(\mu+1)}\left\{1-(1-p^{-1})\sum_{n=0}^{A+\mu}\xi_p(\Pi_p)^{-2n}p^{(2s-1)n}\right\} \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)}\Bigg[1-\xi_p(\Pi_p)^{2A+2\mu+2}p^{(-2s+1)(A+\mu+1)} \\ & \times\left\{1-(1-p^{-1})\frac{1-\xi_p(\Pi_p)^{-2(A+\mu+1)}p^{(2s-1)(A+\mu+1)}}{1-\xi_p(\Pi_p)^{-2}p^{2s-1}}\right\}\Bigg] \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)}\left(1-\xi_p(\Pi_p)^{2}p^{-2s}\right) \\ & \times\left(1-\xi_p(\Pi_p)^{2}p^{-2s+1}\right)^{-1}\left(1-\xi_p(\Pi_p)^{2(A+\mu+1)}p^{(-2s+1)(A+\mu+1)}\right) \\ = \ & \xi_p(\Pi_p)^{-A}p^{A(s-1)}\left(1-\xi_p(\Pi_p)^{2}p^{-2s}\right)R_p(m,A;s). \end{aligned}$$

Hence we see that

$$I_p^{(m)}(\boldsymbol{d}(a_p);s)=\begin{cases}\xi_p(\Pi_p)^{-A}p^{A(s-1)}\left(1-p^{-2s+1}\right)^{-1}\left(1-p^{-2s}\right) & (m=0),\\ \xi_p(\Pi_p)^{-A}p^{A(s-1)}\left(1-p^{-2s}\right)R_p(m,A;s) & (m\in p^{-A}\mathbf{Z}_p),\\ 0 & (m\notin p^{-A}\mathbf{Z}_p),\end{cases}$$

where $A=\mathrm{ord}_K\, a_p$ and $\mu=\mathrm{ord}_p\, m$.

Suppose that p splits. For $a_p\in K_p^{\times}$, put $A=\mathrm{ord}_K\, a_p=a_1+a_2$ if $a_p\in\Pi_{p,1}^{a_1}\Pi_{p,2}^{a_2}\mathcal{O}_{K,p}^{\times}$. It is easily seen that

$$\begin{aligned} & I_p^{(m)}(\boldsymbol{d}(a_p);s) \\ = \ & \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{As}\int_{p^A\mathbf{Z}_p}\psi_p(-mx)dx \\ & +\xi_p(\Pi_{p,1})^{a_1}\xi_p(\Pi_{p,2})^{a_2}p^{-As}\sum_{n=-\infty}^{A-1}\xi_p(p)^{-n}p^{(2s-1)n}\int_{\mathbf{Z}_p^{\times}}\psi_p(-mp^nx)dx. \end{aligned}$$

If $m=0$, we have

$$\begin{aligned}
&I_p^{(0)}(\boldsymbol{d}(a_p);s)\\
&= \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\\
&\quad \times\left\{1+(1-p^{-1})\xi_p(p)^A p^{-A(2s-1)}\sum_{n=-A+1}^{\infty}\xi_p(p)^n p^{(-2s+1)n}\right\}\\
&= \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\left\{1+(1-p^{-1})\xi_p(p)p^{-2s+1}\sum_{n=0}^{\infty}\xi_p(p)^n p^{(-2s+1)n}\right\}\\
&= \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\left(1-\xi_p(p)p^{-2s+1}\right)^{-1}\left(1-\xi_p(p)p^{-2s}\right).
\end{aligned}$$

If $m\notin p^{-A}\mathbf{Z}_p$, we obtain

$$I_p^{(m)}(\boldsymbol{d}(a_p);s)=0.$$

If $m\in p^{-A}\mathbf{Z}_p$, we have

$$\begin{aligned}
&I_p^{(m)}(\boldsymbol{d}(a_p);s)\\
&= \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\\
&\quad \times\left[1-\xi_p(p)^{A+\mu+1}p^{(-2s+1)(A+\mu+1)}\left\{1-(1-p^{-1})\sum_{n=0}^{A+\mu}\xi_p(p)^{-n}p^{(2s-1)n}\right\}\right]\\
&= \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\left(1-\xi_p(p)p^{-2s}\right)\\
&\quad \times\left(1-\xi_p(p)p^{-2s+1}\right)^{-1}\left(1-\xi_p(p)^{A+\mu+1}p^{(-2s+1)(A+\mu+1)}\right)\\
&= \xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\left(1-\xi_p(p)p^{-2s}\right)R_p(m,A;s).
\end{aligned}$$

Hence we see that

$$\begin{aligned}
&I_p^{(m)}(\boldsymbol{d}(a_p);s)\\
&= \begin{cases}
\xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\left(1-p^{-2s+1}\right)^{-1}\left(1-p^{-2s}\right) & (m=0),\\
\xi_p(\Pi_{p,1})^{-a_2}\xi_p(\Pi_{p,2})^{-a_1}p^{A(s-1)}\left(1-p^{-2s}\right)R_p(m,A;s) & (m\in p^{-A}\mathbf{Z}_p),\\
0 & (m\notin p^{-A}\mathbf{Z}_p),
\end{cases}
\end{aligned}$$

where $A=\mathrm{ord}_K a_p=a_1+a_2$ $(a_p\in\Pi_{p,1}^{a_1}\Pi_{p,2}^{a_2}\mathcal{O}_{K,p}^{\times})$ and $\mu=\mathrm{ord}_p m$.

Let $v=\infty$. The Iwasawa decomposition of $S_\infty\boldsymbol{n}(x)_\infty\boldsymbol{d}(a_\infty)=\boldsymbol{n}(X)\boldsymbol{d}(Y)u_\infty$, where

$$X=-\frac{x}{x^2+\mathrm{N}(a_\infty)^2},\quad Y=\frac{a_\infty}{x+\mathrm{N}(a_\infty)i},$$
$$u_\infty=(x+\mathrm{N}(a_\infty)i)^{-1}\begin{pmatrix}-x & \mathrm{N}(a_\infty)\\ -\mathrm{N}(a_\infty) & -x\end{pmatrix}$$

implies that

$$\begin{aligned}
I_\infty^{(m)}(\boldsymbol{d}(a_\infty);s) &= (-1)^{\kappa+2}a_\infty^{s-\kappa/2-1}\overline{a_\infty}^{s+\kappa/2+1}\\
&\quad\times\int_{\mathbf{R}}e^{-2\pi imx}(x+\mathrm{N}(a_\infty)i)^{-s-\kappa/2-1}(x-\mathrm{N}(a_\infty)i)^{-s+\kappa/2+1}dx.
\end{aligned}$$

For $y>0$, $\alpha,\beta\in\mathbf{C}$ ($\mathrm{Re}(\alpha+\beta)>1$) and $m\in\mathbf{R}$, it is known that

$$\int_{\mathbf{R}} e^{-2\pi imx}(x+iy)^{-\alpha}(x-iy)^{-\beta}dx$$
$$=\begin{cases} \dfrac{i^{\beta-\alpha}2^{\alpha+\beta}\pi^{\alpha+\beta}m^{\alpha+\beta-1}}{e^{2\pi ym}\Gamma(\alpha)\Gamma(\beta)}\Phi(4\pi ym;\alpha,\beta) & (m>0), \\ i^{\beta-\alpha}2^{2-\alpha-\beta}\pi y^{1-\alpha-\beta}\dfrac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} & (m=0), \\ \dfrac{i^{\beta-\alpha}2^{\alpha+\beta}\pi^{\alpha+\beta}\left|m\right|^{\alpha+\beta-1}}{e^{2\pi y|m|}\Gamma(\alpha)\Gamma(\beta)}\Phi(4\pi y\left|m\right|;\beta,\alpha) & (m<0) \end{cases}$$

(cf. [Miy]), where

$$\Phi(z;A,B)=\int_0^\infty e^{-zu}(u+1)^{A-1}u^{B-1}du.$$

Using the notation of the classical Whittaker function $W_{\nu,\mu}(z)$, we get

$$\Phi(z;A,B)=e^{z/2}z^{-(A+B)/2}\Gamma(B)W_{(A-B)/2,(A+B-1)/2}(z).$$

Hence we obtain

$$\int_{\mathbf{R}} e^{-2\pi imx}(x+iy)^{-\alpha}(x-iy)^{-\beta}dx$$
$$=\begin{cases} i^{\beta-\alpha}\pi^{(\alpha+\beta)/2}y^{-(\alpha+\beta)/2}m^{(\alpha+\beta-2)/2}\Gamma(\alpha)^{-1} & \\ \quad\times W_{(\alpha-\beta)/2,(\alpha+\beta-1)/2}(4\pi ym) & (m>0), \\ i^{\beta-\alpha}2^{2-\alpha-\beta}\pi y^{1-\alpha-\beta}\Gamma(\alpha+\beta-1)\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} & (m=0), \\ i^{\beta-\alpha}\pi^{(\alpha+\beta)/2}y^{-(\alpha+\beta)/2}\left|m\right|^{(\alpha+\beta-2)/2}\Gamma(\beta)^{-1} & \\ \quad\times W_{(\beta-\alpha)/2,(\alpha+\beta-1)/2}(4\pi y\left|m\right|) & (m<0). \end{cases}$$

This equation implies that

$$I_\infty^{(m)}(\boldsymbol{d}(a_\infty);s)$$
$$= i^{\kappa+2}a_\infty^{-\kappa/2-1}\overline{a_\infty}^{\kappa/2+1}$$
$$\times\begin{cases} \pi^s m^{s-1}\Gamma(s+\kappa/2+1)^{-1}W_{\kappa/2+1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)m) & (m>0), \\ a_\infty^{1-s}\overline{a_\infty}^{1-s}2^{2-2s}\pi\dfrac{\Gamma(2s-1)}{\Gamma(s+\kappa/2+1)\Gamma(s-\kappa/2-1)} & (m=0), \\ \pi^s\left|m\right|^{s-1}\Gamma(s-\kappa/2-1)^{-1}W_{-\kappa/2-1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)\left|m\right|) & (m<0). \end{cases}$$

Note that $\Gamma(s+\kappa/2+1)=P_{\kappa/2}(s)\Gamma(s)$ and $\Gamma(s-\kappa/2-1)=\prod_{j=-1}^{\kappa/2-1}(s-\kappa/2+j)^{-1}\Gamma(s)=(-1)^{\kappa/2+1}P_{\kappa/2}(1-s)^{-1}\Gamma(s)$ since $\Gamma(X+1)=X\Gamma(X)$.

We now complete the calculation of $E^{(m)}_{\kappa+2}(h,\Xi;s)$. If $m=0$, then we have

$$\begin{aligned} I^{(0)}(\boldsymbol{d}(a);s) &= \xi(a^\sigma)^{-1}\,\|a\|_{\mathbf{A}}^{1-s}\,i^{\kappa+2}2^{2-2s}\pi\zeta(2s-1)\zeta(2s)^{-1} \\ &\quad\times\Gamma(2s-1)\Gamma(s+\kappa/2+1)^{-1}\Gamma(s-\kappa/2-1)^{-1} \\ &= \xi(a)\,\|a\|_{\mathbf{A}}^{1-s}\,G(2(1-s))P_{\kappa/2}(1-s)G(2s)^{-1}P_{\kappa/2}(s)^{-1}. \end{aligned}$$

Here we use the fact $\Gamma(X/2)\Gamma((X+1)/2)=2^{1-X}\sqrt{\pi}\Gamma(X)$. Hence we obtain

$$\begin{aligned} E^{(0)}_{\kappa+2}(h,\Xi;s) &= \phi_{\kappa+2}(h;s)+I^{(0)}(h;s) \\ &= \xi(a)\,\|a\|_{\mathbf{A}}^{s}\,J(u_\infty,i)^{-\kappa-2}+J(u_\infty,i)^{-\kappa-2}I^{(0)}(\boldsymbol{d}(a);s) \\ &= G(2s)^{-1}P_{\kappa/2}(s)^{-1}\xi(a)J(u_\infty,i)^{-\kappa-2} \\ &\quad\times\left\{\|a\|_{\mathbf{A}}^{s}\,G(2s)P_{\kappa/2}(s)+\|a\|_{\mathbf{A}}^{1-s}\,G(2(1-s))P_{\kappa/2}(1-s)\right\}. \end{aligned}$$

Suppose that $m>0$. If $(m\,\mathrm{N}(a))_p\in\mathbf{Z}_p$ for each finite prime p, we have

$$\begin{aligned} &I^{(m)}(\boldsymbol{d}(a);s) \\ &= i^{\kappa+2}\pi^s m^{s-1}\xi(a^\sigma)^{-1}\Gamma(s+\kappa/2+1)^{-1}\zeta(2s)^{-1} \\ &\quad\times W_{\kappa/2+1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)m)\prod_{p<\infty}p^{\mathrm{ord}_p\mathrm{N}(a_p)(s-1)}\prod_{p<\infty}R_p(m,\mathrm{ord}_p\mathrm{N}(a_p);s) \\ &= G(2s)^{-1}P_{\kappa/2}(s)^{-1}i^{\kappa+2}\xi(a)\,\|a\|_{\mathbf{A}}^{1-s}\,|\mathrm{N}(a_\infty)m|^{s-1} \\ &\quad\times W_{\kappa/2+1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)m)\prod_{p<\infty}R_p(m,\mathrm{ord}_p\mathrm{N}(a_p);s). \end{aligned}$$

Hence we obtain

$$\begin{aligned} E^{(m)}_{\kappa+2}(h,\Xi;s) &= I^{(m)}(h;s) \\ &= \psi(mb)J(u_\infty,i)^{-\kappa-2}I^{(m)}(\boldsymbol{d}(a);s) \\ &= G(2s)^{-1}P_{\kappa/2}(s)^{-1}\psi(mb)J(u_\infty,i)^{-\kappa-2}i^{\kappa+2}\xi(a)\,\|a\|_{\mathbf{A}}^{1-s}\,|\mathrm{N}(a_\infty)m|^{s-1} \\ &\quad\times W_{\kappa/2+1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)m)\prod_{p<\infty}R_p(m,\mathrm{ord}_p\mathrm{N}(a_p);s). \end{aligned}$$

In a similar way, for $m<0$, we have

$$\begin{aligned} &I^{(m)}(\boldsymbol{d}(a);s) \\ &= i^{\kappa+2}\pi^s\,|m|^{s-1}\,\xi(a^\sigma)^{-1}\Gamma(s-\kappa/2-1)^{-1}\zeta(2s)^{-1} \\ &\quad\times W_{-\kappa/2-1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)\,|m|)\prod_{p<\infty}p^{\mathrm{ord}_p\mathrm{N}(a_p)(s-1)}\prod_{p<\infty}R_p(m,\mathrm{ord}_p\mathrm{N}(a_p);s) \\ &= G(2s)^{-1}P_{\kappa/2}(1-s)\xi(a)\,\|a\|_{\mathbf{A}}^{1-s}\,|\mathrm{N}(a_\infty)m|^{s-1} \\ &\quad\times W_{-\kappa/2-1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)\,|m|)\prod_{p<\infty}R_p(m,\mathrm{ord}_p\mathrm{N}(a_p);s), \end{aligned}$$

if $(m\,\mathrm{N}(a))_p \in \mathbf{Z}_p$ for each finite prime p. Hence we obtain

$$\begin{aligned} E^{(m)}_{\kappa+2}(h,\Xi;s) &= \psi(mb)J(u_\infty,i)^{-\kappa-2}I^{(m)}(\boldsymbol{d}(a);s) \\ &= G(2s)^{-1}\psi(mb)J(u_\infty,i)^{-\kappa-2}P_{\kappa/2}(1-s)\xi(a)\,\|a\|_{\mathbf{A}}^{1-s}\,|\mathrm{N}(a_\infty)m|^{s-1} \\ &\qquad \times W_{-\kappa/2-1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)\,|m|)\prod_{p<\infty}R_p(m,\mathrm{ord}_p\,\mathrm{N}(a_p);s). \end{aligned}$$

Therefore we have the following. For $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$, set

$$E^*_{\kappa+2}(h,\Xi;s) = G(2s)P_{\kappa/2}(s)E_{\kappa+2}(h,\Xi;s).$$

Put

$$C(a) = \{n \in \mathbf{Q} \subset \mathbf{Q}_{\mathbf{A}};\ (n\,\mathrm{N}(a))_p \in \mathbf{Z}_p \ \text{ for all } \ p < \infty\}$$

for $a \in K^\times_{\mathbf{A}}$. Then the Fourier expansion of $E^*_{\kappa+2}(h,\Xi;s)$ is given by

$$E^*_{\kappa+2}(h,\Xi;s) = \sum_{m\in C(a)} e^{(m)}_{\kappa+2}(h,\Xi;s),$$

where

$$\begin{aligned} e^{(0)}_{\kappa+2}(h,\Xi;s) &= \xi(a)J(u_\infty,i)^{-\kappa-2} \\ &\quad \times \left\{\|a\|_{\mathbf{A}}^{s}\,G(2s)P_{\kappa/2}(s) + \|a\|_{\mathbf{A}}^{1-s}\,G(2(1-s))P_{\kappa/2}(1-s)\right\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &e^{(m)}_{\kappa+2}(h,\Xi;s) \\ &\quad = \xi(a)J(u_\infty,i)^{-\kappa-2}\psi(mb)\,\|a\|_{\mathbf{A}}^{1-s}\,|\mathrm{N}(a_\infty)m|^{s-1} \\ &\qquad \times \prod_{p<\infty}R_p(m,\mathrm{ord}_p\,\mathrm{N}(a_p);s) \\ &\qquad \times \begin{cases} i^{\kappa+2}W_{\kappa/2+1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)m) & (m>0), \\ P_{\kappa/2}(s)P_{\kappa/2}(1-s)W_{-\kappa/2-1,s-1/2}(4\pi\,\mathrm{N}(a_\infty)\,|m|) & (m<0). \end{cases} \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we see that $E^*_{\kappa+2}(h,\Xi;s)$ is continued to an entire function of s on $\mathbf{C}$. Note that $E^*_{\kappa+2}(h,\Xi;s)$ has no pole at $s=0$, $s=1$ and $s=1/2$. Since

$$R_p(m,M;1-s) = p^{(M+\mathrm{ord}_p\,m)(2s-1)}R_p(m,M;s)$$

for every finite prime p, we also have a functional equation

$$E^*_{\kappa+2}(h,\Xi;s) = E^*_{\kappa+2}(h,\Xi;1-s).$$

This completes the proof. □

4 Main results

We now state the main results of this paper. For $f \in S_{\ell-1}(D,\chi_0;\chi_0\Omega)$, we put

$$\mathcal{Z}(f,\Xi;s) = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} f(h)E_{k-\ell+2}(h,\Xi;s)\overline{\theta_{\chi_1}(h)}dh.$$

Let $f \in S_{\ell-1}(D,\chi_0;\chi_0\Omega)$ be a Hecke eigenform with eigenvalues $\{\Lambda_p\}$ satisfying $\mathfrak{F}_{D,p}f = \varepsilon_p f$ for each $p \mid D$. We put

$$\boldsymbol{W}_{f,2} = \begin{cases} \chi_{0,2}(\Pi_2)^{-1}W_f(\boldsymbol{d}(\Pi_2^{-1})\overline{\boldsymbol{n}}(2)) & (\mathrm{ord}_2 D = 2,\ \varepsilon_2 = i\chi_{0,2}(\sqrt{D})), \\ \chi_{0,2}(\Pi_2)^{-1}W_f(\boldsymbol{d}(\Pi_2^{-1})\overline{\boldsymbol{n}}(4)) & (\mathrm{ord}_2 D = 3), \\ W_f(I) & (\text{otherwise}) \end{cases} \tag{4.1}$$

and

$$\begin{aligned} \mathfrak{C}_2(f) &= \boldsymbol{W}_{f,2} \\ &\quad \times \begin{cases} \Lambda_2 & (\mathrm{ord}_2 D = 2,\ \varepsilon_2 = i\chi_{0,2}(\sqrt{D})), \\ \left\{\Lambda_2 - \sqrt{2}\varepsilon_2\chi_{0,2}(\sqrt{D})\lambda_{K,2}(\psi_2)^{-1}\right\} & (\mathrm{ord}_2 D = 3), \\ 1 & (\text{otherwise}). \end{cases} \end{aligned} \tag{4.2}$$

Here $\overline{\boldsymbol{n}}(2)$ and $\overline{\boldsymbol{n}}(4)$ are elements of $H_{\mathbf{Q}_2}$.

Theorem 4.1 *Let $f \in S_{\ell-1}(D,\chi_0;\chi_0\Omega)$ be a Hecke eigenform with eigenvalues $\{\Lambda_p\}$ satisfying $\mathfrak{F}_{D,p}f = \varepsilon_p f$ for each $p \mid D$ ($\varepsilon_p = \pm 1$). Let Ξ be a Hecke character of K satisfying (3.1) and (3.2). Then we have*

$$\mathcal{Z}(f,\Xi;s) = \frac{(-1)^{(k-\ell)/2}\pi e^{2\pi}}{(4\pi)^{(k+\ell)/2+s-1}}\boldsymbol{W}_{f,2}\Gamma((k+\ell)/2+s-1)\zeta(2s)^{-1}L(f,\Xi;s)\prod_{p\mid D}D_p(f;s),$$

where

$$\begin{aligned} &D_p(f;s) \\ &= \begin{cases} \varepsilon_p\chi_{0,p}(\sqrt{D})\overline{\lambda_{K,p}(\psi_p)}p^s & (p \neq 2), \\ -p^{2s} & (p = 2,\ \mathrm{ord}_p D = 2, \\ & \quad \varepsilon_p = -i\chi_{0,p}(\sqrt{D})), \\ \Lambda_p p^{2s-2} & (p = 2,\ \mathrm{ord}_p D = 2, \\ & \quad \varepsilon_p = i\chi_{0,p}(\sqrt{D})), \\ \left(\Lambda_p\varepsilon_p\chi_{0,p}(-\sqrt{D})\lambda_{K,p}(\psi_p)p^{-1/2} - 1\right)p^{3s-1/2} & (p = 2,\ \mathrm{ord}_p D = 3). \end{cases} \end{aligned}$$

Remark 4.2 Note that $i\chi_{0,p}(\sqrt{D}) = \pm 1$ for $p = 2$ and $\mathrm{ord}_p D = 2$.

Corollary 4.3 *Let f and Ξ be as in Theorem 4.1. Put*

$$L^*(f, \Xi; s) = (2\pi)^{-2s} |D|^s \,\Gamma((k-\ell)/2 + s + 1)\Gamma((k+\ell)/2 + s - 1)L(f, \Xi; s).$$

If $\mathfrak{C}_2(f) \neq 0$, then $L^*(f, \Xi; s)$ is continued to an entire function of s on $\mathbf{C}$, and satisfies a functional equation

$$L^*(f, \Xi; s) = L^*(f, \Xi; 1-s).$$

The proofs of these results will be given in Section 6.

5 Whittaker functions

5.1 Local Whittaker function

5.1.1 Definitions

Let p be a finite prime of $\mathbf{Q}$. First suppose that $p \nmid D$. For $\Lambda_p \in \mathbf{C}$, set

$$\mathcal{W}_p(\Lambda_p) = \left\{ W \colon H_p \to \mathbf{C}; \begin{array}{l} (1)\ W(t\boldsymbol{n}(x)hu) = (\chi_0\Omega)(t)\psi_p(x)\widetilde{\chi_{0,p}}(u)W(h) \\ \quad (t \in K_p^1,\, h \in H_p,\, x \in \mathbf{Q}_p,\, u \in \mathcal{U}_0(D)_p), \\ (2)\ \mathcal{T}_p W = \Lambda_p W \end{array} \right\}.$$

If p splits in $K/\mathbf{Q}$, we replace the condition (2) above with $\mathcal{T}_{p,j}W = \Lambda_{p,j}W$ $(j = 1, 2)$ for $\Lambda_p = (\Lambda_{p,1}, \Lambda_{p,2}) \in \mathbf{C}^2$. Next suppose that $p \mid D$. For $\Lambda_p \in \mathbf{C}$ and $\varepsilon_p \in \{\pm 1\}$, set

$$\mathcal{W}_p(\Lambda_p, \varepsilon_p) = \left\{ W \colon H_p \to \mathbf{C}; \begin{array}{l} (1)\ W(t\boldsymbol{n}(x)hu) = (\chi_0\Omega)(t)\psi_p(x)\widetilde{\chi_{0,p}}(u)W(h) \\ \quad (t \in K_p^1,\, h \in H_p,\, x \in \mathbf{Q}_p,\, u \in \mathcal{U}_0(D)_p), \\ (2)\ \mathcal{T}_p W = \Lambda_p W, \\ (3)\ W(hw_{D,p}) = \varepsilon_p W(h) \quad (h \in H_p) \end{array} \right\}.$$

We call $\mathcal{W}_p(\Lambda_p)$ (or $\mathcal{W}_p(\Lambda_p, \varepsilon_p)$) the *space of local Whittaker functions.*

5.1.2 Unramified case

First, we study the structure of the space of local Whittaker functions $\mathcal{W}_p(\Lambda_p)$ in the unramified case (inert and split).

Lemma 5.1 *Suppose that p is inert in $K/\mathbf{Q}$. For $W \in \mathcal{W}_p(\Lambda_p)$, we have the following.*

(1) $\operatorname{supp} W \subset \bigcup_{k \geq 0} N_p \boldsymbol{d}(p^k)\mathcal{U}_p$.

(2) *For $k \in \mathbf{Z}$, we have $W(\boldsymbol{d}(p^k)) = \{w(k) + p^{-1}w(k-1)\}\, W(I)$ with*

$$w(n) = \begin{cases} \sum_{r=0}^{n} x_+^{n-r} x_-^r & (n \geq 0), \\ 0 & (n < 0), \end{cases}$$

where $x_\pm$ are the roots of $t^2 - p^{-2}(1-p-\Lambda_p)t + p^{-2} = 0$.

(3) *If $W(I) = 0$, then we have $W \equiv 0$.*

Proof. (1) Since $W(\boldsymbol{d}(p^k)) = W(\boldsymbol{d}(p^k)\boldsymbol{n}(1)) = W(\boldsymbol{n}(p^{2k})\boldsymbol{d}(p^k)) = \psi_p(p^{2k})W(\boldsymbol{d}(p^k))$, we have $W(\boldsymbol{d}(p^k)) = 0$ if $k < 0$.
(2) We set $F(k) = W(\boldsymbol{d}(p^k))$. Since $\mathcal{T}_p W(h) = \Lambda_p W(h)$, we have

$$\Lambda_p F(k) = -F(k-1) - F(k) \sum_{x \in \mathbf{Z}_p^\times / p\mathbf{Z}_p} \psi_p(p^{2k-1}x) - F(k+1) \sum_{y \in \mathbf{Z}_p / p^2\mathbf{Z}_p} \psi_p(p^{2k}y).$$

For $k \geq 0$, we see

$$\sum_{x \in \mathbf{Z}_p^\times / p\mathbf{Z}_p} \psi_p(p^{2k-1}x) = \begin{cases} -1 & (k=0), \\ p-1 & (k \geq 1), \end{cases} \qquad \sum_{y \in \mathbf{Z}_p / p^2\mathbf{Z}_p} \psi_p(p^{2k}y) = p^2.$$

It follows from (1) that

$$\begin{cases} F(1) = p^{-2}(1-\Lambda_p)F(0), \\ p^2 F(k+2) - (1-p-\Lambda_p)F(k+1) + F(k) = 0 \quad (k \geq 0). \end{cases}$$

Hence we get

$$\begin{aligned} F(k) &= \frac{x_+^k - x_-^k}{x_+ - x_-} F(1) - x_+ x_- \frac{x_+^{k-1} - x_-^{k-1}}{x_+ - x_-} F(0) \\ &= \frac{p^{-2}}{x_+ - x_-} \left\{ (1-\Lambda_p)(x_+^k - x_-^k) - (x_+^{k-1} - x_-^{k-1}) \right\} F(0) \\ &= \left\{ w(k) + p^{-1}w(k-1) \right\} F(0). \end{aligned}$$

Therefore we have

$$W(\boldsymbol{d}(p^k)) = \left\{ w(k) + p^{-1}w(k-1) \right\} W(I)$$

for all $k \in \mathbf{Z}$.
(3) is clear. □

Proposition 5.2 *Suppose that p is inert in $K/\mathbf{Q}$. Then $\dim \mathcal{W}_p(\Lambda_p) = 1$ and we have*

$$\mathcal{W}_p(\Lambda_p) = \mathbf{C} \cdot W_p^0,$$

where W_p^0 is an element of $\mathcal{W}_p(\Lambda_p)$ given by

$$W_p^0(\boldsymbol{n}(x)\boldsymbol{d}(p^k)u) = \psi_p(x)\widetilde{\chi_{0,p}}(u)\left\{w(k) + p^{-1}w(k-1)\right\}$$

for $x \in \mathbf{Q}_p$, $k \in \mathbf{Z}$ and $u \in \mathcal{U}_p$. We have

$$\sum_{k=0}^{\infty} W_p^0(\boldsymbol{d}(p^k))t^k = \left(1 + p^{-1}t\right)\left(1 - (1 - p - \Lambda_p)p^{-2}t + p^{-2}t^2\right)^{-1}$$

as a formal power series.

Proof. The assertions are easily verified. □

Lemma 5.3 *Suppose that p splits in $K/\mathbf{Q}$. For $W \in \mathcal{W}_p(\Lambda_p)$, we have the following.*

(1) $\operatorname{supp} W \subset \bigcup_{\substack{k_1,k_2 \in \mathbf{Z} \\ k_1+k_2 \geq 0}} N_p \boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})\mathcal{U}_p.$

(2) *For $k_1, k_2 \in \mathbf{Z}$, we have*

$$W(\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})) = (\chi_0\Omega)(\Pi_{p,1}^{-1}\Pi_{p,2})^{k_1} w_1(k_1 + k_2)W(I)$$

with

$$w_1(n) = \begin{cases} \displaystyle\sum_{r=0}^{n} x_+^{n-r}x_-^r & (n \geq 0), \\ 0 & (n < 0), \end{cases}$$

where $x_\pm$ are the roots of $t^2 - p^{-1}\chi_{0,p}(\Pi_{p,1})\Lambda_{p,1}t + p^{-1}(\chi_0\Omega)(\Pi_{p,1}/\Pi_{p,2}) = 0$.

(3) *If $W(I) = 0$, then we have $W \equiv 0$.*

Proof. (1) Since $W(\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})) = W(\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})\boldsymbol{n}(1)) = W(\boldsymbol{n}(p^{k_1+k_2})\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})) = \psi_p(p^{k_1+k_2})W(\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2}))$, we have $W(\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})) = 0$ if $k_1 + k_2 < 0$.
(2) We set $F(m,n) = W(\boldsymbol{d}(\Pi_{p,1}^{m}\Pi_{p,2}^{n}))$. From the assumption, it follows that

$$\Lambda_{p,1}F(m,n) = \chi_{0,p}(\Pi_{p,1}^{-1})F(m-1,n) + \chi_{0,p}(\Pi_{p,1}^{-1})F(m,n+1)\Psi(m+n) \tag{5.1}$$

and

$$F(m,n) = (\chi_0\Omega)(\Pi_{p,1}^{-1}\Pi_{p,2})F(m-1,n+1), \tag{5.2}$$

where

$$\Psi(N) = \sum_{x\in \mathbf{Z}_p/p\mathbf{Z}_p} \psi_p(p^N x) = \begin{cases} p & (N\ge 0), \\ 0 & (N=-1). \end{cases}$$

From (5.1) and (5.2), we get

$$\Lambda_{p,1}F(m,n) = \chi_{0,p}(\Pi_{p,2}^{-1})\Omega(\Pi_{p,1}\Pi_{p,2}^{-1})F(m,n-1) + \chi_{0,p}(\Pi_{p,1}^{-1})F(m,n+1)\Psi(m+n).$$

This equation implies a recursion formula

$$\begin{cases} \Lambda_{p,1}F(m,-m) = \chi_{0,p}(\Pi_{p,1}^{-1})pF(m,-m+1), \\ \chi_{0,p}(\Pi_{p,1}^{-1})pF(m,n+2) - \Lambda_{p,1}F(m,n+1) \\ \qquad +\chi_{0,p}(\Pi_{p,2}^{-1})\Omega(\Pi_{p,1}\Pi_{p,2}^{-1})F(m,n) = 0 \quad (n\ge -m). \end{cases}$$

Hence we have

$$\begin{aligned} F(m,n) &= \left\{\chi_{0,p}(\Pi_{p,1})p^{-1}\Lambda_{p,1}\frac{x_+^{m+n}-x_-^{m+n}}{x_+-x_-} - x_+x_-\frac{x_+^{m+n-1}-x_-^{m+n-1}}{x_+-x_-}\right\}F(m,-m) \\ &= w_1(m+n)F(m,-m) \end{aligned}$$

for $n\ge -m$. Since

$$F(m,-m) = (\chi_0\Omega)(\Pi_{p,1}^{-1}\Pi_{p,2})^m F(0,0)$$

by (5.2), we obtain

$$F(m,n) = (\chi_0\Omega)(\Pi_{p,1}^{-1}\Pi_{p,2})^m w_1(m+n)F(0,0)$$

for $m+n\ge 0$, which proves (2). The third assertion of the lemma is clear. □

Proposition 5.4 *Suppose that p splits in $K/\mathbf{Q}$. Then $\dim \mathcal{W}_p(\Lambda_p)=1$ and we have*

$$\mathcal{W}_p(\Lambda_p) = \mathbf{C}\cdot W_p^0,$$

where W_p^0 is an element of $\mathcal{W}_p(\Lambda_p)$ given by

$$W_p^0(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2})u) = \psi_p(x)\widetilde{\chi_{0,p}}(u)(\chi_0\Omega)(\Pi_{p,1}^{-1}\Pi_{p,2})^{k_1}w_1(k_1+k_2)$$

for $x\in\mathbf{Q}_p$, $k_1,k_2\in\mathbf{Z}$ and $u\in\mathcal{U}_p$. We have

$$\begin{aligned} &\sum_{k_1,k_2=0}^{\infty} W_p^0(\boldsymbol{d}(\Pi_{p,1}^{k_1}\Pi_{p,2}^{k_2}))t_1^{k_1}t_2^{k_2} \\ &= \left(1-p^{-1}t_1t_2\right)\prod_{j=1,2}\left(1-\Lambda_{p,j}\chi_{0,p}(\Pi_{p,j}^{\sigma})\Omega(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma})p^{-1}t_j + (\chi_0\Omega)(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma})p^{-1}t_j^2\right)^{-1} \end{aligned}$$

as a formal power series.

Proof. The assertions are easily verified. □

5.1.3 Ramified case

We next study the structure of the space of local Whittaker functions $\mathcal{W}_p(\Lambda_p, \varepsilon_p)$ in the ramified case. Note that $(\chi_1\xi)(y) = (\chi_0\Xi)(y)$ for $y \in K_p^\times$ in this case. When p ramifies in $K/\mathbf{Q}$, we put

$$w(n) = \begin{cases} \sum_{r=0}^{n} x_+^{n-r} x_-^r & (n \geq 0), \\ 0 & (n < 0), \end{cases}$$

where $x_\pm$ are the roots of $t^2 - p^{-1}\chi_{0,p}(\Pi_p)^{-1}\Lambda_p t + p^{-1}\chi_{0,p}(\Pi_p)^{-2} = 0$. Set $\pi_p = \mathrm{N}(\Pi_p)$. Note that $\pi_p \in p\mathbf{Z}_p^\times$. When $\mathrm{ord}_p D = 1$, we put

$$\begin{aligned} A_p &= \chi_{0,p}(\sqrt{D}) \sum_{x \in \mathbf{Z}_p^\times / p\mathbf{Z}_p} \psi_p(D^{-1}x^{-1})\omega_p(x) \\ &= \chi_{0,p}(\sqrt{D})\sqrt{p}\,\overline{\lambda_{K,p}(\psi_p)}. \end{aligned} \tag{5.3}$$

The last equation follows from $\lambda_{K,p}(\psi_p) = \sqrt{p}^{-1} \sum_{a \in \mathbf{Z}_p^\times / p\mathbf{Z}_p} \psi_p(p^{-1}a)\omega_p(p^{-1}a)$ obtained by Lemma 3.2 (2). Note that $A_p = \pm\sqrt{p}$.

Lemma 5.5 *Suppose that p ramifies in $K/\mathbf{Q}$ and $p \neq 2$. For $W \in \mathcal{W}_p(\Lambda_p, \varepsilon_p)$, we have the following.*

(1) $\mathrm{supp}\, W \subset \bigcup_{k \geq 0} N_p \boldsymbol{d}(\Pi_p^k)\mathcal{U}_0(D)_p \cup \bigcup_{k \geq 0} N_p \boldsymbol{d}(\Pi_p^k) w_{D,p} \mathcal{U}_0(D)_p.$

(2) *For $k \in \mathbf{Z}$, we have*

$$W(\boldsymbol{d}(\Pi_p^k)) = \left\{ w(k) - \chi_{0,p}(\Pi_p)^{-1} p^{-1} \varepsilon_p A_p w(k-1) \right\} W(I)$$

and

$$W(\boldsymbol{d}(\Pi_p^k) w_{D,p}) = \varepsilon_p W(\boldsymbol{d}(\Pi_p^k)).$$

(3) *If $W(I) = 0$, then we have $W \equiv 0$.*

Proof. Since $D \in p\mathbf{Z}_p^\times$ in this case, we have

$$H_p = P_p\mathcal{U}_0(D)_p \cup P_p S_p \mathcal{U}_0(D)_p.$$

(1) Since $W(\boldsymbol{d}(\Pi_p^k)) = W(\boldsymbol{d}(\Pi_p^k)\boldsymbol{n}(1)) = W(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^k))$, we have $W(\boldsymbol{d}(\Pi_p^k)) = 0$ if $k < 0$. It is clear that $W(\boldsymbol{d}(\Pi_p^k) w_{D,p}) = 0$ if $k < 0$.

(2) Put $F_0(k)=W(\boldsymbol{d}(\Pi_p^k))$ and $F_{-1}(k)=W(\boldsymbol{d}(\Pi_p^k)S_p)$. From the assumption, we have

$$\begin{aligned}
&\Lambda_p W(\boldsymbol{d}(\Pi_p^k))\\
&= \chi_{0,p}(\Pi_p)^{-1}\sum_{x\in\mathbf{Z}_p/p\mathbf{Z}_p} W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1}))+\chi_{0,p}(\Pi_p)\sum_{y\in\mathbf{Z}_p/p\mathbf{Z}_p} W(\boldsymbol{d}(\Pi_p^k)\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p))\\
&= \chi_{0,p}(\Pi_p)\sum_{y\in\mathbf{Z}_p/p\mathbf{Z}_p} W(\boldsymbol{n}(\pi_p^k y)\boldsymbol{d}(\Pi_p^{k+1}))+\chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1}))\\
&\quad +\chi_{0,p}(\Pi_p)^{-1}\sum_{x\in\mathbf{Z}_p^\times/p\mathbf{Z}_p} W\left(\boldsymbol{n}(\pi_p^k D^{-1}x^{-1})\boldsymbol{d}(\Pi_p^{k-1})S_p\begin{pmatrix}-\pi_p^{-1}Dx & -1\\ & -\pi_p D^{-1}x^{-1}\end{pmatrix}\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
\Lambda_p F_0(k) &= \chi_{0,p}(\Pi_p)F_0(k+1)\sum_{y\in\mathbf{Z}_p/p\mathbf{Z}_p}\psi_p(\pi_p^k y)+\chi_{0,p}(\Pi_p)^{-1}F_0(k-1)\\
&\quad +\chi_{0,p}(\Pi_p)^{-1}F_{-1}(k-1)\sum_{x\in\mathbf{Z}_p^\times/p\mathbf{Z}_p}\psi_p(\pi_p^k D^{-1}x^{-1})\omega_p(-\pi_p^{-1}Dx).
\end{aligned}$$

By the equations $W(\boldsymbol{d}(\Pi_p^k)w_{D,p})=\varepsilon_p W(\boldsymbol{d}(\Pi_p^k))$ and $w_{D,p}=\boldsymbol{d}(\Pi_p^{-1})S_p\boldsymbol{d}((\Pi_p^\sigma)^{-1}\sqrt{D})$, we obtain

$$\chi_{0,p}(-\Pi_p^{-1}\sqrt{D})F_{-1}(k-1)=\varepsilon_p F_0(k).$$

Hence we have

$$\begin{aligned}
\Lambda_p F_0(k) &= \chi_{0,p}(\Pi_p)F_0(k+1)\sum_{y\in\mathbf{Z}_p/p\mathbf{Z}_p}\psi_p(\pi_p^k y)+\chi_{0,p}(\Pi_p)^{-1}F_0(k-1)\\
&\quad +\chi_{0,p}(-\sqrt{D})^{-1}\varepsilon_p F_0(k)\sum_{x\in\mathbf{Z}_p^\times/p\mathbf{Z}_p}\psi_p(\pi_p^k D^{-1}x^{-1})\omega_p(x).
\end{aligned}$$

Since

$$\sum_{x\in\mathbf{Z}_p^\times/p\mathbf{Z}_p}\psi_p(\pi_p^k D^{-1}x^{-1})\omega_p(x)=\begin{cases}\displaystyle\sum_{x\in\mathbf{Z}_p^\times/p\mathbf{Z}_p}\omega_p(x)=0 & (k\geq 1),\\ \displaystyle\sum_{x\in\mathbf{Z}_p^\times/p\mathbf{Z}_p}\psi_p(D^{-1}x^{-1})\omega_p(x)=\chi_{0,p}(\sqrt{D})^{-1}A_p & (k=0),\end{cases}$$

we get

$$\begin{cases}F_0(1)=-\chi_{0,p}(\Pi_p)^{-1}p^{-1}\left\{\varepsilon_p A_p-\Lambda_p\right\}F_0(0),\\ \chi_{0,p}(\Pi_p)pF_0(k+2)-\Lambda_p F_0(k+1)+\chi_{0,p}(\Pi_p)^{-1}F_0(k)=0 \quad (k\geq 0).\end{cases}$$

Thus

$$\begin{aligned} F_0(k) &= \frac{x_+^k - x_-^k}{x_+ - x_-} F_0(1) - x_+ x_- \frac{x_+^{k-1} - x_-^{k-1}}{x_+ - x_-} F_0(0) \\ &= \left\{ w(k) - \chi_{0,p}(\Pi_p)^{-1} p^{-1} \varepsilon_p A_p w(k-1) \right\} F_0(0). \end{aligned}$$

Therefore we have

$$W(\boldsymbol{d}(\Pi_p^k)) = \left\{ w(k) - \chi_{0,p}(\Pi_p)^{-1} p^{-1} \varepsilon_p A_p w(k-1) \right\} W(I)$$

for all $k \in \mathbf{Z}$.

(3) is clear. □

Proposition 5.6 *Suppose that p ramifies in $K/\mathbf{Q}$ and $p \neq 2$. Then* $\dim \mathcal{W}_p(\Lambda_p, \varepsilon_p) = 1$ *and we have*

$$\mathcal{W}_p(\Lambda_p, \varepsilon_p) = \mathbf{C} \cdot W^0_{p,\varepsilon_p},$$

where W^0_{p,ε_p} is an element of $\mathcal{W}_p(\Lambda_p, \varepsilon_p)$ given by

$$W^0_{p,\varepsilon_p}(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p^k)u) = \psi_p(x)\widetilde{\chi_{0,p}}(u) \left\{ w(k) - \chi_{0,p}(\Pi_p)^{-1} p^{-1} \varepsilon_p A_p w(k-1) \right\}$$

and

$$W^0_{p,\varepsilon_p}(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p^k) w_{D,p} u) = \varepsilon_p \psi_p(x) \widetilde{\chi_{0,p}}(u) W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^k))$$

for $x \in \mathbf{Q}_p$, $k \in \mathbf{Z}$ and $u \in \mathcal{U}_0(D)_p$. We have

$$\begin{aligned} &\sum_{k=0}^{\infty} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^k)) t^k \\ &= \left(1 - \chi_{0,p}(\Pi_p)^{-1} p^{-1} \varepsilon_p A_p t\right) \left(1 - \Lambda_p \chi_{0,p}(\Pi_p)^{-1} p^{-1} t + \chi_{0,p}(\Pi_p)^{-2} p^{-1} t^2\right)^{-1} \end{aligned}$$

as a formal power series.

Proof. The assertions are easily verified. □

Lemma 5.7 *Suppose that $p = 2$ and* $\mathrm{ord}_p D = 2$. *For $W \in \mathcal{W}_p(\Lambda_p, \varepsilon_p)$, we have the following.*

(1) *We have*

$$\operatorname{supp} W \subset \bigcup_{k\geq 0} N_p \boldsymbol{d}(\Pi_p^k)\mathcal{U}_0(D)_p \cup N_p \boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p)_p\mathcal{U}_0(D)_p \cup \bigcup_{k\geq 0} N_p \boldsymbol{d}(\Pi_p^k) w_{D,p}\mathcal{U}_0(D)_p.$$

(2) *If* $\varepsilon_p = -i\chi_{0,p}(\sqrt{D})$, *then we have*

$$\begin{cases} W(\boldsymbol{d}(\Pi_p^k)) = w(k)W(I), \\ W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)) = 0, \\ W(\boldsymbol{d}(\Pi_p^k)w_{D,p}) = \varepsilon_p w(k)W(I) \end{cases}$$

for $k \in \mathbf{Z}$. *If* $\varepsilon_p = i\chi_{0,p}(\sqrt{D})$, *then we have*

$$\begin{cases} W(\boldsymbol{d}(\Pi_p^k)) = \left\{ \dfrac{\Lambda_p}{p^2\chi_{0,p}(\Pi_p)} w(k) - \dfrac{1}{p\chi_{0,p}(\Pi_p)^2} w(k-1) \right\} W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p)), \\ W(\boldsymbol{d}(\Pi_p^k)w_{D,p}) = \varepsilon_p W(\boldsymbol{d}(\Pi_p^k)) \end{cases}$$

for $k \in \mathbf{Z}$.

(3) *If* $\varepsilon_p = -i\chi_{0,p}(\sqrt{D})$ *and* $W(I) = 0$, *then we have* $W \equiv 0$. *If* $\varepsilon_p = i\chi_{0,p}(\sqrt{D})$ *and* $W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p)) = 0$, *then we have* $W \equiv 0$.

Proof. In this case, we have $\omega_p|_{1+p\mathbf{Z}_p^\times} = -\mathbf{1}$, $\omega_p|_{1+p^2\mathbf{Z}_p} = \mathbf{1}$ and

$$H_p = P_p\mathcal{U}_0(D)_p \cup P_p\overline{\boldsymbol{n}}(p)\mathcal{U}_0(D)_p \cup P_pS_p\mathcal{U}_0(D)_p.$$

(1) Since $W(\boldsymbol{d}(\Pi_p^k)) = W(\boldsymbol{d}(\Pi_p^k)\boldsymbol{n}(1)) = W(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^k)) = \psi_p(\pi_p^k)W(\boldsymbol{d}(\Pi_p^k))$, we have $W(\boldsymbol{d}(\Pi_p^k)) = 0$ if $k < 0$. Thus we also have $W(\boldsymbol{d}(\Pi_p^k)w_{D,p}) = 0$ if $k < 0$. We see that

$$W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)) = -W\left(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)\begin{pmatrix} 1+p & 1 \\ -p^2 & 1-p \end{pmatrix}\right) = -W(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)).$$

Hence we have $W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)) = 0$ if $k \neq -1$.

(2) We put $F_0(k) = W(\boldsymbol{d}(\Pi_p^k))$, $F_2(k) = W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p))$ and $F_{-1}(k) = W(\boldsymbol{d}(\Pi_p^k)S_p)$. It is easily seen that

$$\chi_{0,p}(-\Pi_p^{-2}\sqrt{D})F_{-1}(k-2) = \varepsilon_p F_0(k), \tag{5.4}$$

since $w_{D,p} = \boldsymbol{d}(\Pi_p^{-2})S_p\boldsymbol{d}((\Pi_p^\sigma)^{-2}\sqrt{D})$. Put $\pi_p = p\alpha$ ($\alpha \in \mathbf{Z}_p^\times$). Then $\psi_p(p^{-1}\pi_p^{-1})\chi_{0,p}(p^{-1}) = \psi_p(p^{-2}\alpha^{-1})\omega_p(\alpha^{-1}) = -i$. Hence, from the equation

$$W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)w_{D,p}) = W\left(\boldsymbol{n}(p^{-1}\pi_p^k)\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)\begin{pmatrix} -p^{-1}\sqrt{D} & \\ p\sqrt{D} & p\sqrt{D}^{-1} \end{pmatrix}\right),$$

we have

$$\varepsilon_p F_2(-1) = -i\chi_{0,p}(-\sqrt{D})F_2(-1) = i\chi_{0,p}(\sqrt{D})F_2(-1). \tag{5.5}$$

It follows that $F_2(-1) = 0$ if $\varepsilon_p = -i\chi_{0,p}(\sqrt{D})$. Now, by the assumption, we obtain

$$\begin{aligned}
&\Lambda_p W(\boldsymbol{d}(\Pi_p^k)) \\
&= \chi_{0,p}(\Pi_p)^{-1}\sum_{x=0,1} W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1})) + \chi_{0,p}(\Pi_p)\sum_{y=0,1} W(\boldsymbol{d}(\Pi_p^k)\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p)) \\
&= \chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})) + \chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})\overline{\boldsymbol{n}}(p)\overline{\boldsymbol{n}}(D\pi_p^{-1}-p)) \\
&\quad + \chi_{0,p}(\Pi_p)W(\boldsymbol{d}(\Pi_p^{k+1})) + \chi_{0,p}(\Pi_p)W(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^{k+1})) \\
&= \chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})) + \chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})\overline{\boldsymbol{n}}(p)) \\
&\quad + \chi_{0,p}(\Pi_p)W(\boldsymbol{d}(\Pi_p^{k+1})) + \chi_{0,p}(\Pi_p)\psi_p(\pi_p^k)W(\boldsymbol{d}(\Pi_p^{k+1})),
\end{aligned}$$

which implies

$$\begin{aligned}
\Lambda_p F_0(k) &= \chi_{0,p}(\Pi_p)^{-1}F_0(k-1) + \chi_{0,p}(\Pi_p)^{-1}F_2(k-1) \\
&\quad + \chi_{0,p}(\Pi_p)\left\{1+\psi_p(\pi_p^k)\right\}F_0(k+1).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\Lambda_p W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)) \\
&= \chi_{0,p}(\Pi_p)^{-1}\sum_{x=0,1} W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1})) \\
&\quad + \chi_{0,p}(\Pi_p)\sum_{y=0,1} W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p)) \\
&= \chi_{0,p}(\Pi_p)^{-1}W\left(\boldsymbol{n}(p^{-1}\pi_p^k)\boldsymbol{d}(\Pi_p^{k-1})S_p\begin{pmatrix}-p\pi_p^{-1} & -1 \\ & -p^{-1}\pi_p\end{pmatrix}\right) \\
&\quad + \chi_{0,p}(\Pi_p)^{-1}W\left(\boldsymbol{n}(-p^{-1}\pi_p^k)\boldsymbol{d}(\Pi_p^{k-1})S_p\begin{pmatrix}-\pi_p^{-1}(p+D) & -1 \\ p+p^{-1}D & p^{-1}\pi_p\end{pmatrix}\right) \\
&\quad + \chi_{0,p}(\Pi_p)W(\boldsymbol{d}(\Pi_p^{k+1})\overline{\boldsymbol{n}}(p\pi_p)) + \chi_{0,p}(\Pi_p)W\left(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^{k+1})\begin{pmatrix}1-p & -p\pi_p^{-1} \\ p\pi_p & 1+p\end{pmatrix}\right) \\
&= \chi_{0,p}(-\Pi_p p)^{-1}\psi_p(p^{-1}\pi_p^k)W\left(\boldsymbol{d}(\Pi_p^{k-1})S_p\right) \\
&\quad + \chi_{0,p}(-\Pi_p(p+D))^{-1}\psi_p(-p^{-1}\pi_p^k)W\left(\boldsymbol{d}(\Pi_p^{k-1})S_p\right) \\
&\quad + \chi_{0,p}(\Pi_p)W(\boldsymbol{d}(\Pi_p^{k+1})) + \chi_{0,p}(\Pi_p(1-p))\psi_p(\pi_p^k)W\left(\boldsymbol{d}(\Pi_p^{k+1})\right),
\end{aligned}$$

which implies

$$\begin{aligned}
\Lambda_p F_2(-1) &= \chi_{0,p}(-\Pi_p p)^{-1}\left\{\psi_p(p^{-1}\pi_p^{-1}) + \omega_p(1+p^{-1}D)\overline{\psi_p(p^{-1}\pi_p^{-1})}\right\}F_{-1}(-2) \\
&\quad + \chi_{0,p}(\Pi_p)\left\{1+\omega_p(1-p)\psi_p(\pi_p^{-1})\right\}F_0(0).
\end{aligned}$$

We see that $\omega_p(1-p)=\omega_p(1+p^{-1}D)=-1$ and $\psi_p(\pi_p^{-1})=-1$. Put $\pi_p=p\alpha$ ($\alpha\in\mathbf{Z}_p^\times$). Then $\omega_p(p)=\omega_p(\alpha)$. On the other hand, $\psi_p(p^{-1}\pi_p^{-1})=\psi_p(p^{-2}\alpha^{-1})=-\omega_p(\alpha)i$. Thus

$$\chi_{0,p}(p)^{-1}\left\{\psi_p(p^{-1}\pi_p^{-1})-\overline{\psi_p(p^{-1}\pi_p^{-1})}\right\}=\omega_p(p)\left\{-\omega_p(\alpha)i-\omega_p(\alpha)i\right\}=-2i.$$

Hence we get the following:

$$\Lambda_pF_0(0) = \chi_{0,p}(\Pi_p)^{-1}F_2(-1)+2\chi_{0,p}(\Pi_p)F_0(1), \tag{5.6}$$
$$\Lambda_pF_0(k+1) = \chi_{0,p}(\Pi_p)^{-1}F_0(k)+2\chi_{0,p}(\Pi_p)F_0(k+2) \quad (k\geq 0), \tag{5.7}$$
$$\Lambda_pF_2(-1) = 2\chi_{0,p}(\Pi_p)F_0(0)-2i\chi_{0,p}(-\Pi_p)^{-1}F_{-1}(-2). \tag{5.8}$$

We obtain

$$\begin{aligned}F_0(k) &= \frac{x_+^k-x_-^k}{x_+-x_-}F_0(1)-x_+x_-\frac{x_+^{k-1}-x_-^{k-1}}{x_+-x_-}F_0(0)\\ &= \frac{x_+^{k+1}-x_-^{k+1}}{x_+-x_-}F_0(0)-x_+x_-\frac{x_+^k-x_-^k}{x_+-x_-}F_2(-1) \quad (k\geq 0)\end{aligned}$$

from (5.6) and (5.7). By (5.4) and (5.8), we get

$$\Lambda_pF_2(-1)=2\chi_{0,p}(\Pi_p)\left\{1-i\varepsilon_p\chi_{0,p}(\sqrt{D})^{-1}\right\}F_0(0).$$

Therefore we have

$$\begin{cases}F_0(k)=w(k)F_0(0)-\dfrac{1}{2\chi_{0,p}(\Pi_p)^2}w(k-1)F_2(-1) & (k\geq 0),\\ F_{-1}(k)=-\chi_{0,p}(\Pi_p^{-2}\sqrt{D})^{-1}\varepsilon_pF_0(k+2) & (k\in\mathbf{Z}),\\ \left\{\varepsilon_p-i\chi_{0,p}(\sqrt{D})\right\}F_2(-1)=0, & \\ \Lambda_pF_2(-1)=2\chi_{0,p}(\Pi_p)\left\{1-i\varepsilon_p\chi_{0,p}(\sqrt{D})^{-1}\right\}F_0(0). & \end{cases}$$

Note that $\varepsilon_p=\pm i\chi_{0,p}(\sqrt{D})$. If $\varepsilon_p=-i\chi_{0,p}(\sqrt{D})$, then we have

$$\begin{cases}W(\boldsymbol{d}(\Pi_p^k))=w(k)W(I),\\ W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p))=0\end{cases}$$

for all $k\in\mathbf{Z}$. If $\varepsilon_p=i\chi_{0,p}(\sqrt{D})$, then we have

$$W(\boldsymbol{d}(\Pi_p^k))=\left\{\frac{\Lambda_p}{p^2\chi_{0,p}(\Pi_p)}w(k)-\frac{1}{p\chi_{0,p}(\Pi_p)^2}w(k-1)\right\}W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p))$$

for all $k\in\mathbf{Z}$.

(3) is clear. □

Proposition 5.8 *Suppose that* $p = 2$ *and* $\operatorname{ord}_p D = 2$. *Then* $\dim \mathcal{W}_p(\Lambda_p, \varepsilon_p) = 1$ *and we have*

$$\mathcal{W}_p(\Lambda_p, \varepsilon_p) = \mathbf{C} \cdot W^0_{p,\varepsilon_p},$$

where W^0_{p,ε_p} *is an element of* $\mathcal{W}_p(\Lambda_p, \varepsilon_p)$ *given as follows:*

(1) *If* $\varepsilon_p = -i\chi_{0,p}(\sqrt{D})$, *then*

$$W^0_{p,\varepsilon_p}(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p^k)hu) = \psi_p(x)\widetilde{\chi_{0,p}}(u) \times \begin{cases} w(k) & (h = I), \\ 0 & (h = \overline{\boldsymbol{n}}(p)), \\ -i\chi_{0,p}(\sqrt{D})w(k) & (h = w_{D,p}) \end{cases}$$

for $x \in \mathbf{Q}_p$, $k \in \mathbf{Z}$ *and* $u \in \mathcal{U}_0(D)_p$. *We have*

$$\sum_{k=0}^{\infty} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^k))t^k = \left(1 - \Lambda_p\chi_{0,p}(\Pi_p)^{-1}p^{-1}t + \chi_{0,p}(\Pi_p)^{-2}p^{-1}t^2\right)^{-1}$$

as a formal power series.

(2) *If* $\varepsilon_p = i\chi_{0,p}(\sqrt{D})$, *then*

$$\begin{aligned} & W^0_{p,\varepsilon_p}(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p^k)hu) \\ = \ & \psi_p(x)\widetilde{\chi_{0,p}}(u) \\ & \times \begin{cases} \dfrac{\Lambda_p}{p^2}w(k) - \dfrac{1}{p\chi_{0,p}(\Pi_p)}w(k-1) & (h = I), \\ \chi_{0,p}(\Pi_p) & (h = \overline{\boldsymbol{n}}(p),\ k = -1), \\ i\chi_{0,p}(\sqrt{D})\left\{\dfrac{\Lambda_p}{p^2}w(k) - \dfrac{1}{p\chi_{0,p}(\Pi_p)}w(k-1)\right\} & (h = w_{D,p}) \end{cases} \end{aligned}$$

for $x \in \mathbf{Q}_p$, $k \in \mathbf{Z}$ *and* $u \in \mathcal{U}_0(D)_p$. *We have*

$$\begin{aligned} \sum_{k=0}^{\infty} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^k))t^k \ &= \ p^{-2}\left(\Lambda_p - p\chi_{0,p}(\Pi_p)^{-1}t\right) \\ & \quad \times \left(1 - \Lambda_p\chi_{0,p}(\Pi_p)^{-1}p^{-1}t + \chi_{0,p}(\Pi_p)^{-2}p^{-1}t^2\right)^{-1} \end{aligned}$$

as a formal power series.

Proof. The assertions are easily verified. □

Lemma 5.9 *Suppose that $p=2$ and $\mathrm{ord}_p D=3$. For $W\in\mathcal{W}_p(\Lambda_p,\varepsilon_p)$, we have the following.*

(1) *We have*

$$\begin{aligned}\mathrm{supp}\,W \quad\subset\quad & \bigcup_{k\geq 0} N_p\boldsymbol{d}(\Pi_p^k)\mathcal{U}_0(D)_p\cup N_p\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p)_p\mathcal{U}_0(D)_p\\ &\cup N_p\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)_p\mathcal{U}_0(D)_p\cup N_p\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p)_p\mathcal{U}_0(D)_p\\ &\cup\bigcup_{k\geq 0} N_p\boldsymbol{d}(\Pi_p^k)w_{D,p}\mathcal{U}_0(D)_p.\end{aligned}$$

(2) *For $k\in\mathbf{Z}$, we have*

$$\begin{cases} W(\boldsymbol{d}(\Pi_p^k))=\left\{\dfrac{C_p(\Lambda_p)}{p\chi_{0,p}(\Pi_p)}w(k)-\dfrac{1}{p\chi_{0,p}(\Pi_p)^2}w(k-1)\right\}W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)),\\ W(\boldsymbol{d}(\Pi_p^k)w_{D,p})=\varepsilon_p W(\boldsymbol{d}(\Pi_p^k)),\\ W(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p))=\varepsilon_p B_p W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)),\\ W(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p))=-i\omega_p(1+p)\varepsilon_p B_p W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)),\end{cases}$$

where

$$B_p=e^{-\pi i/4}\chi_{0,p}(-p^{-1}\Pi_p\sqrt{D}) \tag{5.9}$$

and

$$C_p(\Lambda_p)=\Lambda_p-\chi_{0,p}(\Pi_p)^{-1}\varepsilon_p B_p(1-i\omega_p(1+p)). \tag{5.10}$$

(3) *If $W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2))=0$, then we have $W\equiv 0$.*

Proof. In this case, we have $\omega_p|_{1+p^2\mathbf{Z}_p^\times}=-\mathbf{1}$, $\omega_p|_{1+p^3\mathbf{Z}_p}=\mathbf{1}$ and

$$H_p=P_p\mathcal{U}_0(D)_p\cup P_p\overline{\boldsymbol{n}}(p)\mathcal{U}_0(D)_p\cup P_p\overline{\boldsymbol{n}}(p^2)\mathcal{U}_0(D)_p\cup P_p\overline{\boldsymbol{n}}(p^2+p)\mathcal{U}_0(D)_p\cup P_pS_p\mathcal{U}_0(D)_p.$$

(1) Since $W(\boldsymbol{d}(\Pi_p^k))=W(\boldsymbol{d}(\Pi_p^k)\boldsymbol{n}(1))=W(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^k))$, we have $W(\boldsymbol{d}(\Pi_p^k))=0$ if $k<0$. We also have $W(\boldsymbol{d}(\Pi_p^k)w_{D,p})=0$ if $k<0$. For $M,X\in\mathbf{Z}_p$ satisfying $M^2X\in D\mathbf{Z}_p$, we see that

$$\begin{aligned}W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(M)) &= \omega_p(1+MX)W\left(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(M)\begin{pmatrix}1+MX & X\\ -M^2X & 1-MX\end{pmatrix}\right)\\ &= \omega_p(1+MX)W(\boldsymbol{n}(\pi_p^kX)\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(M)).\end{aligned}$$

Then we have $W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(M))=0$ if $\psi_p(\pi_p^k X)\neq\omega_p(1+MX)$. This implies the following:

$$\begin{aligned} k\neq -2 &\implies W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p))=0,\\ k\neq -2 &\implies W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2+p))=0,\\ k\neq -1 &\implies W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2))=0.\end{aligned}$$

(2) We put $F_0(k)=W(\boldsymbol{d}(\Pi_p^k))$, $F_2(k)=W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p))$, $F_4(k)=W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2))$, $F_6(k)=W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2+p))$ and $F_{-1}(k)=W(\boldsymbol{d}(\Pi_p^k)S_p)$. It is easily seen that

$$\chi_{0,p}(-\Pi_p^{-3}\sqrt{D})F_{-1}(k-3)=\varepsilon_p F_0(k). \tag{5.11}$$

We obtain

$$\psi_p(p^{-1}\pi_p^{-2})\chi_{0,p}(-p^{-1}(\Pi_p^\sigma)^{-1}\sqrt{D})F_4(-1)=\varepsilon_p F_2(-2) \tag{5.12}$$

from the equation

$$W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p)w_{D,p})=W\left(\boldsymbol{n}(p^{-1}\pi_p^k)\boldsymbol{d}(\Pi_p^{k+1})\overline{\boldsymbol{n}}(p^2)\begin{pmatrix} -p^{-1}(\Pi_p^\sigma)^{-1}\sqrt{D} & \\ (\Pi_p^\sigma)^{-1}\sqrt{D}(p+\pi_p) & p\Pi_p\sqrt{D}^{-1}\end{pmatrix}\right),$$

and we get

$$\psi_p((p^2+p)^{-1}\pi_p^{-2})\chi_{0,p}(-(\Pi_p^\sigma)^{-1}\sqrt{D}(p^2+p)^{-1})F_4(-1)=\varepsilon_p F_6(-2) \tag{5.13}$$

from the equation

$$\begin{aligned}&W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2+p)w_{D,p})\\ &= W\left(\boldsymbol{n}((p^2+p)^{-1}\pi_p^k)\boldsymbol{d}(\Pi_p^{k+1})\overline{\boldsymbol{n}}(p^2)\begin{pmatrix} -(\Pi_p^\sigma)^{-1}\sqrt{D}(p^2+p)^{-1} & \\ \Pi_p\sqrt{D}(p+1)^{-1}(1+p+p\pi_p^{-1}) & \Pi_p\sqrt{D}^{-1}(p^2+p)\end{pmatrix}\right)\end{aligned}$$

respectively. By the assumption, we have

$$\begin{aligned}&\Lambda_p W(\boldsymbol{d}(\Pi_p^k))\\ &= \chi_{0,p}(\Pi_p)^{-1}\sum_{x=0,1}W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1}))+\chi_{0,p}(\Pi_p)\sum_{y=0,1}W(\boldsymbol{d}(\Pi_p^k)\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p))\\ &= \chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1}))+\chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})\overline{\boldsymbol{n}}(p^2)\overline{\boldsymbol{n}}(D\pi_p^{-1}-p^2))\\ &\quad +\chi_{0,p}(\Pi_p)W(\boldsymbol{d}(\Pi_p^{k+1}))+\chi_{0,p}(\Pi_p)W(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^{k+1})).\end{aligned}$$

This implies that

$$\begin{aligned}\Lambda_p F_0(k) &= \chi_{0,p}(\Pi_p)^{-1}F_0(k-1)+\chi_{0,p}(\Pi_p)^{-1}F_4(k-1)\\ &\quad +\chi_{0,p}(\Pi_p)\left\{1+\psi_p(\pi_p^k)\right\}F_0(k+1).\end{aligned} \tag{5.14}$$

Similarly, we obtain

$$\begin{aligned}\Lambda_p F_4(k) &= \chi_{0,p}(\Pi_p)\left\{1+\omega_p(1-p^2)\psi_p(\pi_p^k)\right\}F_0(k+1)\\ &\quad +\chi_{0,p}(\Pi_p)^{-1}F_2(k-1)+\chi_{0,p}(\Pi_p)^{-1}F_6(k-1)\end{aligned} \tag{5.15}$$

from the equation

$$\begin{aligned}&\Lambda_p W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2))\\ &= \chi_{0,p}(\Pi_p)^{-1}\sum_{x=0,1}W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2)\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1}))\\ &\quad +\chi_{0,p}(\Pi_p)\sum_{y=0,1}W(\boldsymbol{d}(\Pi_p^k)\overline{\boldsymbol{n}}(p^2)\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p))\\ &= \chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})\overline{\boldsymbol{n}}(p^2\pi_p^{-1}))+\chi_{0,p}(\Pi_p)^{-1}W(\boldsymbol{d}(\Pi_p^{k-1})\overline{\boldsymbol{n}}(\pi_p^{-1}(p^2+D)))\\ &\quad +\chi_{0,p}(\Pi_p)W(\boldsymbol{d}(\Pi_p^{k+1})\overline{\boldsymbol{n}}(p^2\pi_p))+\chi_{0,p}(\Pi_p)W\left(\boldsymbol{n}(\pi_p^k)\boldsymbol{d}(\Pi_p^{k+1})\begin{pmatrix}1-p^2 & -p^2\pi_p^{-1}\\ p^2\pi_p & 1+p^2\end{pmatrix}\right).\end{aligned}$$

Since $\pi_p \in p\mathbf{Z}_p^\times$ and $(1+p)^{-1} \in 1+p+p^3\mathbf{Z}_p$, we obtain $\pi_p^{-2} \in p^{-2}(1+p^3\mathbf{Z}_p)$, $\psi_p(p^{-1}\pi_p^{-2}) = e^{-\pi i/4}$ and $\psi_p((p^2+p)^{-1}\pi_p^{-2}) = -ie^{-\pi i/4}$. Hence we have

$$\begin{cases}F_{-1}(k)=\varepsilon_p\chi_{0,p}(-\Pi_p^{-3}\sqrt{D})^{-1}F_0(k+3) \quad (k\in\mathbf{Z}),\\ F_2(-2)=\varepsilon_p e^{-\pi i/4}\chi_{0,p}(-p^{-1}\Pi_p\sqrt{D})F_4(-1),\\ F_6(-2)=-i\varepsilon_p e^{-\pi i/4}\chi_{0,p}(-\Pi_p\sqrt{D}(p^2+p)^{-1})F_4(-1),\\ F_0(0)=2^{-1}\chi_{0,p}(\Pi_p)^{-1}\left\{\Lambda_p-\varepsilon_p e^{-\pi i/4}\chi_{0,p}(-p^{-1}\sqrt{D})\left(1-i\omega_p(1+p)\right)\right\}F_4(-1),\\ 2\chi_{0,p}(\Pi_p)F_0(1)=\Lambda_pF_0(0)-\chi_{0,p}(\Pi_p)^{-1}F_4(-1),\\ 2\chi_{0,p}(\Pi_p)F_0(k+2)-\Lambda_pF_0(k+1)+\chi_{0,p}(\Pi_p)^{-1}F_0(k)=0 \quad (k\geq 0).\end{cases}$$

Since

$$W(I)=2^{-1}\chi_{0,p}(\Pi_p)^{-1}C_p(\Lambda_p)W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2))$$

and

$$W(\boldsymbol{d}(\Pi_p))=4^{-1}\chi_{0,p}(\Pi_p)^{-2}\left\{\Lambda_pC_p(\Lambda_p)-p\right\}W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)),$$

we obtain

$$\begin{aligned}W(\boldsymbol{d}(\Pi_p^k)) &= \frac{x_+^k-x_-^k}{x_+-x_-}W(\boldsymbol{d}(\Pi_p))-x_+x_-\frac{x_+^{k-1}-x_-^{k-1}}{x_+-x_-}W(I)\\ &= \left\{\frac{C_p(\Lambda_p)}{2\chi_{0,p}(\Pi_p)}w(k)-\frac{1}{2\chi_{0,p}(\Pi_p)^2}w(k-1)\right\}W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)).\end{aligned}$$

Therefore we have

$$\begin{cases} W(\boldsymbol{d}(\Pi_p^k)) = \left\{ \dfrac{C_p(\Lambda_p)}{p\chi_{0,p}(\Pi_p)} w(k) - \dfrac{1}{p\chi_{0,p}(\Pi_p)^2} w(k-1) \right\} W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)) \quad (k \in \mathbf{Z}), \\ W(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p)) = \varepsilon_p B_p W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)), \\ W(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p)) = -i\omega_p(1+p)\varepsilon_p B_p W(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)). \end{cases}$$

(3) is clear. □

Proposition 5.10 *Suppose that* $p = 2$ *and* $\mathrm{ord}_p D = 3$. *Then* $\dim \mathcal{W}_p(\Lambda_p, \varepsilon_p) = 1$ *and we have*

$$\mathcal{W}_p(\Lambda_p, \varepsilon_p) = \mathbf{C} \cdot W^0_{p,\varepsilon_p},$$

where W^0_{p,ε_p} *is an element of* $\mathcal{W}_p(\Lambda_p, \varepsilon_p)$ *given by*

$$\begin{aligned} &W^0_{p,\varepsilon_p}(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p^k)hu) \\ &= \psi_p(x)\widetilde{\chi_{0,p}}(u) \\ &\quad \times \begin{cases} \dfrac{C_p(\Lambda_p)}{p} w(k) - \dfrac{1}{p\chi_{0,p}(\Pi_p)} w(k-1) & (h = I), \\ \varepsilon_p \left\{ \dfrac{C_p(\Lambda_p)}{p} w(k) - \dfrac{1}{p\chi_{0,p}(\Pi_p)} w(k-1) \right\} & (h = w_{D,p}), \\ \varepsilon_p \chi_{0,p}(\Pi_p) B_p & (h = \overline{\boldsymbol{n}}(p),\ k = -2), \\ -i\omega_p(1+p)\varepsilon_p \chi_{0,p}(\Pi_p) B_p & (h = \overline{\boldsymbol{n}}(p^2+p),\ k = -2), \\ \chi_{0,p}(\Pi_p) & (h = \overline{\boldsymbol{n}}(p^2),\ k = -1) \end{cases} \end{aligned}$$

for $x \in \mathbf{Q}_p$, $k \in \mathbf{Z}$ *and* $u \in \mathcal{U}_0(D)_p$. *We have*

$$\begin{aligned} \sum_{k=0}^{\infty} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^k))t^k &= p^{-1}\left\{C_p(\Lambda_p) - \chi_{0,p}(\Pi_p)^{-1}t\right\} \\ &\quad \times \left(1 - \Lambda_p\chi_{0,p}(\Pi_p)^{-1}p^{-1}t + \chi_{0,p}(\Pi_p)^{-2}p^{-1}t^2\right)^{-1} \end{aligned}$$

as a formal power series.

Proof. The assertions are easily verified. □

5.2 Global Whittaker function

In this subsection, we study some properties of the global Whittaker function W_f attached to $f \in S_{\ell-1}(D, \chi_0; \chi_0\Omega)$.

Proposition 5.11 *Let $f \in S_{\ell-1}(D,\chi_0;\chi_0\Omega)$. For every $h_f \in H_{\mathbf{A},f}$, we have*

$$W_f(\boldsymbol{d}(r)_\infty h_f) = r^{\ell-1}e^{2\pi(1-r^2)}W_f(h_f) \quad (r \in \mathbf{R}_+).$$

Proof. For $h_f \in H_{\mathbf{A},f}$, set $f_{\mathrm{dm},h_f}(h_\infty\langle i\rangle) = J(h_\infty,i)^{\ell-1}f(h_\infty h_f)$. Then $f_{\mathrm{dm},h_f}(h_\infty\langle i\rangle)$ is holomorphic on $\mathfrak{H}$. We have

$$\begin{aligned} &W_f(\boldsymbol{d}(r)_\infty h_f) \\ &= \int_{\mathbf{Q}\backslash\mathbf{Q}_\mathbf{A}} \psi(-x)f(\boldsymbol{n}(x)\boldsymbol{d}(r)_\infty h_f)dx \\ &= \int_{\mathbf{Q}\backslash\mathbf{Q}_\mathbf{A}} \psi(-x)J(\boldsymbol{n}(x_\infty)\boldsymbol{d}(r)_\infty,i)^{1-\ell}f_{\mathrm{dm},\boldsymbol{n}(x_f)h_f}(\boldsymbol{n}(x_\infty)\boldsymbol{d}(r)_\infty\langle i\rangle)dx. \end{aligned}$$

Now, for every $h_f \in H_{\mathbf{A},f}$, we can take $\mathrm{N}(h_f) \in \mathbf{Z}_+$ such that $h_f^{-1}\boldsymbol{n}(\mathrm{N}(h_f)_f)h_f \in \mathcal{U}_0(D)_f$ and $\widetilde{\chi_0}(h_f^{-1}\boldsymbol{n}(\mathrm{N}(h_f)_f)h_f) = 1$. Hence we obtain

$$\begin{aligned} &f_{\mathrm{dm},\boldsymbol{n}(x_f)h_f}(\boldsymbol{n}(\mathrm{N}(h_f)_\infty)h_\infty\langle i\rangle) \\ &= J(\boldsymbol{n}(\mathrm{N}(h_f)_\infty)h_\infty,i)^{\ell-1}f(\boldsymbol{n}(\mathrm{N}(h_f)_\infty)h_\infty\boldsymbol{n}(x_f)h_f) \\ &= J(h_\infty,i)^{\ell-1}f(h_\infty\boldsymbol{n}(-\mathrm{N}(h_f)_f)\boldsymbol{n}(x_f)h_f) \\ &= J(h_\infty,i)^{\ell-1}f(h_\infty\boldsymbol{n}(x_f)h_f) \\ &= f_{\mathrm{dm},\boldsymbol{n}(x_f)h_f}(h_\infty\langle i\rangle). \end{aligned}$$

Thus the function $f_{\mathrm{dm},\boldsymbol{n}(x_f)h_f}$ has a period $\mathrm{N}(h_f) \in \mathbf{Z}_+$. Therefore we have the Fourier expansion

$$f_{\mathrm{dm},\boldsymbol{n}(x_f)h_f}(h_\infty\langle i\rangle) = \sum_{m\in\mathbf{Z}} c(f,\boldsymbol{n}(x_f)h_f;m)\exp\left(\frac{2\pi i h_\infty\langle i\rangle}{\mathrm{N}(h_f)}m\right).$$

From this, we obtain

$$\begin{aligned} &W_f(\boldsymbol{d}(r)_\infty h_f) \\ &= \frac{r^{\ell-1}}{\mathrm{N}(h_f)}\int_0^{\mathrm{N}(h_f)} dx_\infty \int_{\mathbf{Z}_f} dx_f\, \psi(-x)\sum_{m\in\mathbf{Z}} c(f,\boldsymbol{n}(x_f)h_f;m)\exp\left(\frac{2\pi i(x_\infty + r^2 i)}{\mathrm{N}(h_f)}m\right) \\ &= \frac{r^{\ell-1}}{\mathrm{N}(h_f)}\sum_{m\in\mathbf{Z}}\exp\left(-\frac{2\pi r^2}{\mathrm{N}(h_f)}m\right) \\ &\quad \times \int_0^{\mathrm{N}(h_f)} dx_\infty \int_{\mathbf{Z}_f} dx_f\, \exp\left(2\pi i x_\infty\left(\frac{m}{\mathrm{N}(h_f)} - 1\right)\right)\psi_f(-x_f)c(f,\boldsymbol{n}(x_f)h_f;m), \end{aligned}$$

where $\psi_f = \prod_{p<\infty}\psi_p$. Since

$$\int_0^{\mathrm{N}(h_f)} \exp\left(2\pi i x_\infty\left(\frac{m}{\mathrm{N}(h_f)} - 1\right)\right)dx_\infty = \begin{cases} \mathrm{N}(h_f) & (m = \mathrm{N}(h_f)), \\ 0 & (m \neq \mathrm{N}(h_f)), \end{cases}$$

we get

$$W_f(\boldsymbol{d}(r)_\infty h_f) = r^{\ell-1} e^{-2\pi r^2} \int_{\mathbf{Z}_f} \psi_f(-x_f) c(f, \boldsymbol{n}(x_f) h_f; \mathrm{N}(h_f)) dx_f.$$

This equation implies

$$\int_{\mathbf{Z}_f} \psi_f(-x_f) c(f, \boldsymbol{n}(x_f) h_f; \mathrm{N}(h_f)) dx_f = e^{2\pi} W_f(h_f),$$

and we have

$$W_f(\boldsymbol{d}(r)_\infty h_f) = r^{\ell-1} e^{2\pi(1-r^2)} W_f(h_f).$$

□

Proposition 5.2, 5.4, 5.6, 5.8, 5.10 and 5.11 imply the following result.

Proposition 5.12 *If $f \in S_{\ell-1}(D, \chi_0; \chi_0 \Omega)$ is a Hecke eigenform with eigenvalues $\{\Lambda_p\}$ satisfying $\mathfrak{F}_{D,p} f = \varepsilon_p f$ for each $p \mid D$, then we have*

$$W_f(h) = J(h_\infty, i)^{1-\ell} e^{2\pi i (h_\infty \langle i \rangle - i)} \boldsymbol{W}_{f,2} \prod_{p \nmid D} W_p^0(h_p) \prod_{q \mid D} W_{q,\varepsilon_q}^0(h_q)$$

for $h = (h_v)_v \in H_{\mathbf{A}}$, where $\boldsymbol{W}_{f,2}$ is defined in (4.1).

6 Proofs of the main results

In this section, we prove Theorem 4.1 and Corollary 4.3.

Lemma 6.1 *Let $f \in S_{\ell-1}(D, \chi_0; \chi_0 \Omega)$ be a Hecke eigenform with eigenvalues $\{\Lambda_p\}$ satisfying $\mathfrak{F}_{D,p} f = \varepsilon_p f$ for each $p \mid D$. Then we have $\mathcal{Z}(f, \Xi; s) = \boldsymbol{W}_{f,2} \prod_{v \leq \infty} \mathcal{Z}_v(f, \Xi; s)$, where*

$$\mathcal{Z}_v(f, \Xi; s) = \begin{cases} e^{2\pi} \displaystyle\int_{\mathbf{C}^\times} (\chi_1 \xi)(y_\infty) \, |\mathrm{N}(y_\infty)|^{s-1/2} \, \overline{y_\infty}^{\ell-1} e^{-2\pi \mathrm{N}(y_\infty)} \overline{\varphi_{0,\infty}(y_\infty)} d^\times y_\infty & (v = \infty), \\ \displaystyle\int_{K_p^\times} (\chi_1 \xi)(y_p) \, |\mathrm{N}(y_p)|_p^{s-1/2} \, W_p^0(\boldsymbol{d}(y_p)) \varphi_{0,p}(y_p) d^\times y_p & (v = p, \, p \nmid D), \\ \displaystyle\int_{K_p^\times} d^\times y_p \int_{\mathcal{U}_p} du_p & \\ \quad (\chi_1 \xi)(y_p) \, |\mathrm{N}(y_p)|_p^{s-1/2} \, W_{p,\varepsilon_p}^0(\boldsymbol{d}(y_p) u_p) \overline{\mathcal{M}_{\chi_1}^T(u_p) \varphi_{0,p}(y_p)} & (v = p, \, p \mid D), \end{cases}$$

and $\boldsymbol{W}_{f,2}$ is defined in (4.1).

Proof. By unfolding the integral, we have

$$
\begin{aligned}
&\mathcal{Z}(f,\Xi;s)\\
&= \int_{P_{\mathbf{Q}}\backslash H_{\mathbf{A}}} \phi_{k-\ell+2}(h;s)f(h)\overline{\theta_{\chi_1}(h)}dh\\
&= \int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} dx \int_{K^\times\backslash K_{\mathbf{A}}^\times} \|y\|_{\mathbf{A}}^{-1} d^\times y \int_{\mathcal{U}_f} du_f \int_{\mathcal{U}_\infty} du_\infty\\
&\quad \xi(y)\|y\|_{\mathbf{A}}^{s} J(u_\infty,i)^{-k+\ell-2} f(\boldsymbol{n}(x)\boldsymbol{d}(y)u_f u_\infty)\sum_{X\in K}\overline{\mathcal{M}_{\chi_1}^T(\boldsymbol{n}(x)\boldsymbol{d}(y)u_f u_\infty)\varphi_0(X)}\\
&= \int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} dx \int_{K^\times\backslash K_{\mathbf{A}}^\times} d^\times y \int_{\mathcal{U}_f} du_f\\
&\quad \sum_{X\in K}(\chi_1\xi)(y)\|y\|_{\mathbf{A}}^{s-1/2}\psi(-x\,\mathrm{N}(X))f(\boldsymbol{n}(x)\boldsymbol{d}(y)u_f)\overline{\mathcal{M}_{\chi_1}^T(u_f)\varphi_0(yX)}.
\end{aligned}
$$

Since $f\in S_{\ell-1}(D,\chi_0;\chi_0\Omega)$, we see that

$$
\begin{aligned}
&\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} \psi(-x\,\mathrm{N}(X))f(\boldsymbol{n}(x)\boldsymbol{d}(y)u_f)\overline{\mathcal{M}_{\chi_1}^T(u_f)\varphi_0(yX)}dx\\
&\quad = \overline{\mathcal{M}_{\chi_1}^T(u_f)\varphi_0(0)}\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} f(\boldsymbol{n}(x)\boldsymbol{d}(y)u_f)dx\\
&\quad = 0
\end{aligned}
$$

for $X=0$. Moreover, for $X\in K^\times$, we obtain

$$
f(\boldsymbol{n}(x)\boldsymbol{d}(y)u_f)=f(\boldsymbol{d}(X)\boldsymbol{n}(x)\boldsymbol{d}(y)u_f)=f(\boldsymbol{n}(x\,\mathrm{N}(X))\boldsymbol{d}(yX)u_f).
$$

Hence we have

$$
\begin{aligned}
\mathcal{Z}(f,\Xi;s) &= \int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} dx \int_{K^\times\backslash K_{\mathbf{A}}^\times} d^\times y \int_{\mathcal{U}_f} du_f\\
&\quad \sum_{X\in K^\times}(\chi_1\xi)(y)\|y\|_{\mathbf{A}}^{s-1/2}\psi(-x)f(\boldsymbol{n}(x)\boldsymbol{d}(yX)u_f)\overline{\mathcal{M}_{\chi_1}^T(u_f)\varphi_0(yX)}\\
&= \int_{K_{\mathbf{A}}^\times} d^\times y \int_{\mathcal{U}_f} du_f(\chi_1\xi)(y)\|y\|_{\mathbf{A}}^{s-1/2} W_f(\boldsymbol{d}(y)u_f)\overline{\mathcal{M}_{\chi_1}^T(u_f)\varphi_0(y)}.
\end{aligned}
$$

Therefore, by Proposition 5.12, we obtain $\mathcal{Z}(f,\Xi;s)=\boldsymbol{W}_{f,2}\prod_{v\le\infty}\mathcal{Z}_v(f,\Xi;s)$, where

$$\mathcal{Z}_v(f,\Xi;s) = \begin{cases} e^{2\pi}\displaystyle\int_{\mathbf{C}^\times}(\chi_1\xi)(y_\infty)\,|\mathrm{N}(y_\infty)|^{s-1/2}\,\overline{y_\infty}^{\ell-1}e^{-2\pi\,\mathrm{N}(y_\infty)}\overline{\varphi_{0,\infty}(y_\infty)}d^\times y_\infty & (v=\infty),\\ \displaystyle\int_{K_p^\times}(\chi_1\xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\,W_p^0(\boldsymbol{d}(y_p))\varphi_{0,p}(y_p)d^\times y_p & (v=p,\ p\nmid D),\\ \displaystyle\int_{K_p^\times}d^\times y_p\int_{\mathcal{U}_p}du_p & \\ \quad(\chi_1\xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\,W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)u_p)\overline{\mathcal{M}^T_{\chi_1}(u_p)\varphi_{0,p}(y_p)} & (v=p,\ p\mid D).\end{cases}$$

□

Lemma 6.2 *For $v=\infty$, we have*

$$\mathcal{Z}_\infty(f,\Xi;s)=\frac{\pi e^{2\pi}}{(4\pi)^{(k+\ell)/2+s-1}}\Gamma((k+\ell)/2+s-1).$$

Proof. Since $(\chi_1\xi)(y_\infty)=y_\infty^{k+\ell-1}\,|y_\infty|^{-k-\ell+1}$ and $\overline{\varphi_{0,\infty}(y_\infty)}=\overline{y_\infty}^k e^{-2\pi|y_\infty|^2}$, we have

$$\mathcal{Z}_\infty(f,\Xi;s)=e^{2\pi}\int_{\mathbf{C}^\times}\mathrm{N}(y_\infty)^{(k+\ell)/2+s-1}e^{-4\pi\,\mathrm{N}(y_\infty)}d^\times y_\infty.$$

Put $y_\infty=re^{i\theta}$ $(r\in\mathbf{R}_+,\ 0\le\theta<2\pi)$. Since $d^\times y_\infty=2^{-1}\,\mathrm{N}(y_\infty)^{-1}dy_\infty=r^{-1}drd\theta$, we obtain

$$\begin{aligned}\mathcal{Z}_\infty(f,\Xi;s) &= 2\pi e^{2\pi}\int_0^\infty (r^2)^{(k+\ell)/2+s-1}e^{-4\pi r^2}r^{-1}dr\\ &= \frac{\pi e^{2\pi}}{(4\pi)^{(k+\ell)/2+s-1}}\int_0^\infty t^{(k+\ell)/2+s-2}e^{-t}dt\\ &= \frac{\pi e^{2\pi}}{(4\pi)^{(k+\ell)/2+s-1}}\Gamma((k+\ell)/2+s-1).\end{aligned}$$

□

Lemma 6.3 *Suppose that p is inert in $K/\mathbf{Q}$. Then we have*

$$\mathcal{Z}_p(f,\Xi;s)=\left(1-\Xi_p(p)p^{-2s}\right)L_p(f,\Xi_p;s).$$

Proof. Since $\omega_p|_{\mathbf{Z}_p^\times} = \mathbf{1}$, we have

$$\begin{aligned}\mathcal{Z}_p(f,\Xi;s) &= \sum_{n=0}^{\infty}\int_{\mathcal{O}_{K,p}^\times}(\chi_1\xi)(p^n y_p)\left|\mathrm{N}(p^n y_p)\right|_p^{s-1/2} W_p^0(\boldsymbol{d}(p^n y_p))d^\times y_p \\ &= \sum_{n=0}^{\infty}(\chi_1\xi)(p^n)p^{-n(2s-1)}W_p^0(\boldsymbol{d}(p^n))\int_{\mathcal{O}_{K,p}^\times}\chi_{1,p}(y_p)\chi_{0,p}(y_p^\sigma)d^\times y_p \\ &= \sum_{n=0}^{\infty}(\chi_1\xi)(p)^n p^{-n(2s-1)}W_p^0(\boldsymbol{d}(p^n)).\end{aligned}$$

Since $(\chi_1\xi)(p) = \omega_p(p)\Xi_p(p) = -\Xi_p(p)$, Proposition 5.2 shows that

$$\begin{aligned}\mathcal{Z}_p(f,\Xi;s) &= \left(1+(\chi_1\xi)(p)p^{-2s}\right)\left(1-(1-p-\Lambda_p)(\chi_1\xi)(p)p^{-2s-1}+(\chi_1\xi)(p)^2p^{-4s}\right)^{-1} \\ &= (1-\Xi_p(p)p^{-2s})L_p(f,\Xi_p;s).\end{aligned}$$

□

Lemma 6.4 *Suppose that p splits in $K/\mathbf{Q}$. Then we have*

$$\mathcal{Z}_p(f,\Xi;s) = (1-\Xi_p(p)p^{-2s})L_p(f,\Xi_p;s).$$

Proof. Since $\omega_p|_{\mathbf{Q}_p^\times} = \mathbf{1}$, we have

$$\begin{aligned}&\mathcal{Z}_p(f,\Xi;s) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_{\mathcal{O}_{K,p}^\times}(\chi_1\xi)(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2}y_p)\left|\mathrm{N}(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2}y_p)\right|_p^{s-1/2} W_p^0(\boldsymbol{d}(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2}y_p))d^\times y_p \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(\chi_1\xi)(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2})p^{-(n_1+n_2)(s-1/2)}W_p^0(\boldsymbol{d}(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2}))\int_{\mathcal{O}_{K,p}^\times}\chi_{1,p}(y_p)\chi_{0,p}(y_p^\sigma)d^\times y_p \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(\chi_1\xi)(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2})p^{-(n_1+n_2)(s-1/2)}W_p^0(\boldsymbol{d}(\Pi_{p,1}^{n_1}\Pi_{p,2}^{n_2})).\end{aligned}$$

Since $(\chi_1\xi)(\Pi_{p,j}) = \chi_{0,p}(\Pi_{p,j})\Omega(\Pi_{p,j}/\Pi_{p,j}^\sigma)\Xi_p(\Pi_{p,j})$, Proposition 5.4 shows that

$$\begin{aligned}&\mathcal{Z}_p(f,\Xi;s) \\ &= (1-(\chi_1\xi)(p)p^{-2s}) \\ &\quad\times\prod_{j=1,2}\left(1-\Lambda_{p,j}(\chi_0^{-1}\chi_1\xi)(\Pi_{p,j})\Omega(\Pi_{p,j}^\sigma/\Pi_{p,j})p^{-s-1/2}+(\chi_1\xi)(\Pi_{p,j})^2(\chi_0\Omega)(\Pi_{p,j}^\sigma/\Pi_{p,j})p^{-2s}\right)^{-1} \\ &= (1-\Xi_p(p)p^{-2s})L_p(f,\Xi_p;s).\end{aligned}$$

□

Lemma 6.5 *Suppose that p ramifies in $K/\mathbf{Q}$ and $p \neq 2$. Then we have*

$$\mathcal{Z}_p(f, \Xi; s) = \varepsilon_p \chi_{0,p}(\sqrt{D}) \overline{\lambda_{K,p}(\psi_p)} \Xi_p(\Pi_p)^{-1} p^s \left(1 - \Xi_p(p) p^{-2s}\right) L_p(f, \Xi_p; s).$$

Proof. We have

$$\mathcal{U}_p = \coprod_{a_p \in \mathbf{Z}_p / p\mathbf{Z}_p} \boldsymbol{n}(a_p) S_p \mathcal{U}_0(D)_p \cup \mathcal{U}_0(D)_p.$$

This implies that

$$\begin{aligned} &\mathcal{Z}_p(f, \Xi; s) \\ &= \int_{K_p^\times} (\chi_0 \Xi)(y_p) \left|\mathrm{N}(y_p)\right|_p^{s-1/2} W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)) \varphi_{0,p}(y_p) d^\times y_p \\ &\quad + \sum_{a_p \in \mathbf{Z}_p / p\mathbf{Z}_p} \int_{K_p^\times} (\chi_0 \Xi)(y_p) \left|\mathrm{N}(y_p)\right|_p^{s-1/2} W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p) \boldsymbol{n}(a_p) S_p) \overline{\mathcal{M}^T_{\chi_1}(\boldsymbol{n}(a_p) S_p) \varphi_{0,p}(y_p)} d^\times y_p. \end{aligned}$$

Since $W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\boldsymbol{n}(a_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(\boldsymbol{n}(a_p)S_p)\varphi_{0,p}(y_p)} = W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(S_p)\varphi_{0,p}(y_p)}$ and $\mathcal{M}^T_{\chi_1}(S_p)\varphi_{0,p}(y_p) = \lambda_{K,p}(\psi_p) p^{-1/2} \varphi_{0,p}(\sqrt{D} y_p)$, we obtain

$$\begin{aligned} &\mathcal{Z}_p(f, \Xi; s) \\ &= \int_{K_p^\times} (\chi_0 \Xi)(y_p) \left|\mathrm{N}(y_p)\right|_p^{s-1/2} W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)) \varphi_{0,p}(y_p) d^\times y_p \\ &\quad + \overline{\lambda_{K,p}(\psi_p)} p^{1/2} \int_{K_p^\times} (\chi_0 \Xi)(y_p) \left|\mathrm{N}(y_p)\right|_p^{s-1/2} W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p) S_p) \varphi_{0,p}(\sqrt{D} y_p) d^\times y_p. \end{aligned}$$

First, we have

$$\begin{aligned} &\int_{K_p^\times} (\chi_0 \Xi)(y_p) \left|\mathrm{N}(y_p)\right|_p^{s-1/2} W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)) \varphi_{0,p}(y_p) d^\times y_p \\ &= \sum_{n=0}^{\infty} \int_{\mathcal{O}_{K,p}^\times} (\chi_0 \Xi)(\Pi_p^n y_p) \left|\mathrm{N}(\Pi_p^n y_p)\right|_p^{s-1/2} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n y_p)) d^\times y_p \\ &= \sum_{n=0}^{\infty} (\chi_0 \Xi)(\Pi_p^n) p^{-n(s-1/2)} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)) \int_{\mathcal{O}_{K,p}^\times} \chi_{0,p}(y_p) \chi_{0,p}(y_p^\sigma) d^\times y_p \\ &= \sum_{n=0}^{\infty} (\chi_0 \Xi)(\Pi_p)^n p^{-n(s-1/2)} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)). \end{aligned}$$

Next, we have

$$\begin{aligned}
&\int_{K_p^\times}(\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\,W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)S_p)\varphi_{0,p}(\sqrt{D}y_p)d^\times y_p\\
&= (\chi_0\Xi)(-\sqrt{D})^{-1}p^{s-1/2}\int_{K_p^\times}(\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\,W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)w_{D,p})\varphi_{0,p}(-y_p)d^\times y_p\\
&= (\chi_0\Xi)(-\sqrt{D})^{-1}p^{s-1/2}\sum_{n=0}^{\infty}\int_{\mathcal{O}_{K,p}^\times}(\chi_0\Xi)(\Pi_p^n y_p)\left|\mathrm{N}(\Pi_p^n y_p)\right|_p^{s-1/2}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n y_p)w_{D,p})d^\times y_p\\
&= (\chi_0\Xi)(-\sqrt{D})^{-1}p^{s-1/2}\\
&\qquad\times\sum_{n=0}^{\infty}(\chi_0\Xi)(\Pi_p^n)p^{-n(s-1/2)}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)w_{D,p})\int_{\mathcal{O}_{K,p}^\times}\chi_{0,p}(y_p)\chi_{0,p}(y_p)^{-1}d^\times y_p\\
&= \varepsilon_p(\chi_0\Xi)(-\sqrt{D})^{-1}p^{s-1/2}\sum_{n=0}^{\infty}(\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\mathcal{Z}_p(f,\Xi;s) &= (1+\overline{\lambda_{K,p}(\psi_p)}\varepsilon_p(\chi_0\Xi)(-\sqrt{D})^{-1}p^s)\sum_{n=0}^{\infty}(\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n))\\
&= \left(1+\varepsilon_p(\chi_0\Xi)(-\sqrt{D})^{-1}\overline{\lambda_{K,p}(\psi_p)}p^s\right)\left(1-\varepsilon_p\chi_{0,p}(\sqrt{D})\Xi_p(\Pi_p)\overline{\lambda_{K,p}(\psi_p)}p^{-s}\right)\\
&\quad\times\left(1-\Lambda_p\Xi_p(\Pi_p)p^{-s-1/2}+\Xi_p(\Pi_p)^2p^{-2s}\right)^{-1}\\
&= \varepsilon_p\chi_{0,p}(\sqrt{D})\overline{\lambda_{K,p}(\psi_p)}\Xi_p(\Pi_p)^{-1}p^s\left(1-\Xi_p(p)p^{-2s}\right)L_p(f,\Xi_p;s).
\end{aligned}$$

by Proposition 5.6. Here note that $\varepsilon_p\chi_{0,p}(\sqrt{D})\overline{\lambda_{K,p}(\psi_p)}=\pm1$. □

We now consider the case of $p=2$. For $A\in\mathbf{Z}_p$ and $y\in K_p^\times$, set

$$I_p(A,y)=\int_{\mathcal{O}_{K,p}}\psi_p\left(\mathrm{Tr}(\sqrt{D}^{-1}yw^\sigma)+AD^{-1}ww^\sigma\right)dw. \tag{6.1}$$

Lemma 6.6 *Let $p=2$ and $p\mid D$.*

(1) *For $\varepsilon\in\mathbf{Z}_p^\times$ and $a\in\mathbf{Q}_p^\times$ $(\alpha=\mathrm{ord}_p\,a)$, we have*

$$\int_{\mathbf{Z}_p}\psi_p(p^{-1}\varepsilon x^2+ax)dx=\begin{cases}1 & (\alpha=-1),\\ 0 & (\alpha\neq-1)\end{cases}$$

and

$$\int_{\mathbf{Z}_p}\psi_p(p^{-2}\varepsilon x^2+ax)dx=\begin{cases}2^{-1}\left\{1+\psi_p(4^{-1}\varepsilon+a)\right\} & (\alpha\geq-1),\\ 0 & (\alpha\leq-2).\end{cases}$$

(2) *Suppose that* $\mathrm{ord}_p D = 2$. *We have*

$$I_p(A,y) = \begin{cases} |D|_p^{1/2}\,\mathrm{char}_{\mathcal{O}_{K,p}}(y) & (\mathrm{ord}_p A \geq 2), \\ |D|_p^{1/2}\,\mathrm{char}_{\Pi_p^{-1}\mathcal{O}_{K,p}^{\times}}(y) & (\mathrm{ord}_p A = 1). \end{cases}$$

(3) *Suppose that* $\mathrm{ord}_p D = 3$. *We have*

$$I_p(A,y) = \begin{cases} |D|_p^{1/2}\,\mathrm{char}_{\mathcal{O}_{K,p}}(y) & (\mathrm{ord}_p A \geq 3), \\ |D|_p^{1/2}\,\mathrm{char}_{\Pi_p^{-1}\mathcal{O}_{K,p}^{\times}}(y) & (\mathrm{ord}_p A = 2), \\ |D|_p^{1/2}\,\mathrm{char}_{p^{-1}\mathcal{O}_{K,p}^{\times}}(y)\cdot 2^{-1}\left\{1+\psi_p(AD^{-1}+\kappa(y))\right\} & (\mathrm{ord}_p A = 1), \end{cases}$$

where $\kappa(y) = \mathrm{Tr}(\sqrt{D}^{-1}y)$.

Proof. If P is a condition, we put $\delta(P) = 1$ if P holds, and $\delta(P) = 0$ otherwise. The first assertion (1) is easily checked.
(2) The assertion in the case $\mathrm{ord}_p A \geq 2$ is easy. Suppose that $\mathrm{ord}_p A = 1$. Put $\theta = 1 + 2^{-1}\sqrt{D}$. Then $\mathcal{O}_{K,p} = \mathbf{Z}_p + \mathbf{Z}_p\theta$ and θ is a prime element of K_p. We have

$$\begin{aligned} &|D|_p^{-1/2}\, I_p(A,y) \\ &= \int_{\mathbf{Z}_p} dx_1 \int_{\mathbf{Z}_p} dx_2\, \psi_p\left(\mathrm{Tr}(\sqrt{D}^{-1}y(x_1+x_2\theta^{\sigma})) + AD^{-1}(x_1^2 + \mathrm{Tr}(\theta x_1 x_2) + \theta\theta^{\sigma}x_2^2)\right) \\ &= \int_{\mathbf{Z}_p} \psi_p\left(AD^{-1}x_1^2 + \mathrm{Tr}(\sqrt{D}^{-1}y)x_1\right) dx_1 \int_{\mathbf{Z}_p} \psi_p\left(\mathrm{Tr}(\sqrt{D}^{-1}\theta^{\sigma}y)x_2\right) dx_2 \\ &= \delta\left(\mathrm{ord}_p \mathrm{Tr}(\sqrt{D}^{-1}y) = -1\right) \times \delta\left(\mathrm{ord}_p \mathrm{Tr}(\sqrt{D}^{-1}\theta^{\sigma}y) \geq 0\right). \end{aligned}$$

Observe that, for $y = y_1 + y_2\theta$ ($y_1, y_2 \in \mathbf{Q}_p^{\times}$), $\mathrm{Tr}(\sqrt{D}^{-1}y) = y_2$ and $\mathrm{Tr}(\sqrt{D}^{-1}\theta^{\sigma}y) = -y_1$. Since

$$\mathrm{ord}_p\, y_2 = -1,\ \ \mathrm{ord}_p\, y_1 \geq 0 \Longleftrightarrow y \in \Pi_p^{-1}\mathcal{O}_{K,p}^{\times},$$

we have proved (2).
(3) In this case, we put $\theta = 2^{-1}\sqrt{D}$. Then $\mathcal{O}_{K,p} = \mathbf{Z}_p + \mathbf{Z}_p\theta$ and θ is a prime element of K_p. The assertion in the case $\mathrm{ord}_p A \geq 3$ is easily verified. If $\mathrm{ord}_p A = 2$, we have

$$\begin{aligned} &|D|_p^{-1/2}\, I_p(A,y) \\ &= \int_{\mathbf{Z}_p} dx_1 \int_{\mathbf{Z}_p} dx_2\, \psi_p\left(\mathrm{Tr}(\sqrt{D}^{-1}y(x_1+x_2\theta^{\sigma})) + AD^{-1}(x_1^2 + \theta\theta^{\sigma}x_2^2)\right) \\ &= \int_{\mathbf{Z}_p} \psi_p\left(AD^{-1}x_1^2 + \mathrm{Tr}(\sqrt{D}^{-1}y)x_1\right) dx_1 \int_{\mathbf{Z}_p} \psi_p\left(\mathrm{Tr}(\sqrt{D}^{-1}\theta^{\sigma}y)x_2\right) dx_2 \\ &= \mathrm{char}_{\Pi_p^{-1}\mathcal{O}_{K,p}^{\times}}(y). \end{aligned}$$

Suppose that $\mathrm{ord}_p A = 1$ and let $y = y_1 + y_2\theta$. Then we have

$$\begin{aligned}
&|D|_p^{-1/2} I_p(A, y) \\
&= \int_{\mathbf{Z}_p} dx_1 \int_{\mathbf{Z}_p} dx_2\, \psi_p \left(\mathrm{Tr}(\sqrt{D}^{-1}(y_1 + y_2\theta)(x_1 - x_2\theta)) + AD^{-1}(x_1^2 - 4^{-1}Dx_2^2) \right) \\
&= \int_{\mathbf{Z}_p} \psi_p \left(AD^{-1}x_1^2 + y_2 x_1 \right) dx_1 \int_{\mathbf{Z}_p} \psi_p \left(-4^{-1}Ax_2^2 - y_1 x_2 \right) dx_2 \\
&= \delta\left(\mathrm{ord}_p y_2 \geq -1\right) \cdot 2^{-1} \left\{ 1 + \psi_p(AD^{-1} + y_2) \right\} \times \delta\left(\mathrm{ord}_p y_1 = -1\right) \\
&= \mathrm{char}_{p^{-1}\mathcal{O}_{K,p}^\times}(y) \cdot 2^{-1} \left\{ 1 + \psi_p(AD^{-1} + y_2) \right\},
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 6.7 *Let $p = 2$. For $A \in \mathbf{Z}_p$ and $y \in K_p^\times$, we have*

$$\mathcal{M}_{\chi_1}^T(\overline{\boldsymbol{n}}(A))\varphi_{0,p}(y) = |D|_p^{-1/2} I_p(A, y).$$

Proof. Since $\overline{\boldsymbol{n}}(A) = -S_p \boldsymbol{n}(-A) S_p$, we have

$$\begin{aligned}
\mathcal{M}_{\chi_1}^T(\overline{\boldsymbol{n}}(A))\varphi_{0,p}(y) &= \mathcal{M}_{\chi_1}^T(-S_p \boldsymbol{n}(-A) S_p)\varphi_{0,p}(y) \\
&= \int_{K_p} \psi_{K_p}(-yw_p^\sigma)\psi_p(-A\,\mathrm{N}(w_p)) \left\{ \int_{K_p} \psi_{K_p}(w_p z_p^\sigma)\varphi_{0,p}(z_p) dz_p \right\} dw_p \\
&= |D|_p^{1/2} \int_{K_p} \psi_{K_p}(-yw_p^\sigma)\psi_p(-A\,\mathrm{N}(w_p))\varphi_{0,p}(\sqrt{D}w_p) dw_p \\
&= |D|_p^{-1/2} \int_{K_p} \psi_{K_p}(\sqrt{D}^{-1} yw_p^\sigma)\psi_p(AD^{-1}\,\mathrm{N}(w_p))\varphi_{0,p}(w_p) dw_p \\
&= |D|_p^{-1/2} \int_{\mathcal{O}_{K,p}} \psi_p \left(\mathrm{Tr}(\sqrt{D}^{-1} yw_p^\sigma) + AD^{-1}\,\mathrm{N}(w_p) \right) dw_p \\
&= |D|_p^{-1/2} I_p(A, y).
\end{aligned}$$

□

Lemma 6.8 *Suppose that $p = 2$ and $\mathrm{ord}_p D = 2$.*

(1) *If $\varepsilon_p = -i\chi_{0,p}(\sqrt{D})$, then we have*

$$\mathcal{Z}_p(f, \Xi; s) = -\Xi_p(p)^{-1} p^{2s} \left(1 - \Xi_p(p)p^{-2s} \right) L_p(f, \Xi_p; s).$$

(2) *If $\varepsilon_p = i\chi_{0,p}(\sqrt{D})$, then we have*

$$\mathcal{Z}_p(f, \Xi; s) = \Lambda_p \Xi_p(p)^{-1} p^{2s-2} \left(1 - \Xi_p(p)p^{-2s} \right) L_p(f, \Xi_p; s).$$

Proof. We have

$$\mathcal{U}_p = \coprod_{a_p \in \mathbf{Z}_p/p^2\mathbf{Z}_p} \boldsymbol{n}(a_p) S_p \mathcal{U}_0(D)_p \cup \mathcal{U}_0(D)_p \cup \overline{\boldsymbol{n}}(p)\mathcal{U}_0(D)_p.$$

This implies that

$$\begin{aligned}
&\mathcal{Z}_p(f,\Xi;s) \\
&= \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p))\varphi_{0,p}(y_p)d^\times y_p \\
&\quad + \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p))\overline{\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p))\varphi_{0,p}(y_p)}d^\times y_p \\
&\quad + \sum_{a_p\in \mathbf{Z}_p/p^2\mathbf{Z}_p} \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\boldsymbol{n}(a_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(\boldsymbol{n}(a_p)S_p)\varphi_{0,p}(y_p)}d^\times y_p.
\end{aligned}$$

From Lemma 6.6 and Lemma 6.7, we obtain

$$\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p))\varphi_{0,p}(y_p) = |D|_p^{-1/2}\, I_p(p,y_p) = \mathrm{char}_{\Pi_p^{-1}\mathcal{O}^\times_{K,p}}(y_p).$$

Since $W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\boldsymbol{n}(a_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(\boldsymbol{n}(a_p)S_p)\varphi_{0,p}(y_p)} = W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(S_p)\varphi_{0,p}(y_p)}$ and $\mathcal{M}^T_{\chi_1}(S_p)\varphi_{0,p}(y_p) = \lambda_{K,p}(\psi_p)p^{-1}\varphi_{0,p}(\sqrt{D}y_p)$, we have

$$\begin{aligned}
&\mathcal{Z}_p(f,\Xi;s) \\
&= \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p))\varphi_{0,p}(y_p)d^\times y_p \\
&\quad + \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p))\,\mathrm{char}_{\Pi_p^{-1}\mathcal{O}^\times_{K,p}}(y_p)d^\times y_p \\
&\quad + \overline{\lambda_{K,p}(\psi_p)}p\int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)S_p)\varphi_{0,p}(\sqrt{D}y_p)d^\times y_p \\
&= \sum_{n=0}^\infty (\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)) \\
&\quad + (\chi_0\Xi)(\Pi_p)^{-1}p^{s-1/2}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p)) \\
&\quad + (\chi_0\Xi)(-\sqrt{D})^{-1}\overline{\lambda_{K,p}(\psi_p)}p^{2s}\sum_{n=0}^\infty (\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)} W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)w_{D,p}).
\end{aligned}$$

First suppose that $\varepsilon_p = -i\chi_{0,p}(\sqrt{D})$. By Proposition 5.8, we have $W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p)) = 0$ ($y_p \in K_p^\times$). Note that $\chi_{0,p}(D) = \chi_{0,p}(-1) = \omega_p(-1) = -1$ in this case. Hence Proposition 5.8 shows that

$$\begin{aligned}
\mathcal{Z}_p(f,\Xi;s) &= \left(1 + i\overline{\lambda_{K,p}(\psi_p)}\Xi_p(-\sqrt{D})^{-1}p^{2s}\right)\sum_{n=0}^\infty (\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n)) \\
&= \left(1 + i\overline{\lambda_{K,p}(\psi_p)}\Xi_p(-\sqrt{D})^{-1}p^{2s}\right)\left(1 - \Lambda_p\Xi_p(\Pi_p)p^{-s-1/2} + \Xi_p(\Pi_p)^2p^{-2s}\right)^{-1}.
\end{aligned}$$

Since $\lambda_{K,2}(\psi_2) = \dfrac{1}{2}\left\{\psi_2\left(\dfrac{1}{4}\right) - \psi_2\left(\dfrac{3}{4}\right)\right\} = -i$ by Lemma 3.2, we have

$$\mathcal{Z}_p(f,\Xi;s) = -\Xi_p(p)^{-1}p^{2s}\left(1-\Xi_p(p)p^{-2s}\right)L_p(f,\Xi_p;s).$$

Next suppose that $\varepsilon_p = i\chi_{0,p}(\sqrt{D})$. We obtain

$$\begin{aligned}
&\mathcal{Z}_p(f,\Xi;s)\\
&= \left(1 - i\overline{\lambda_{K,p}(\psi_p)}\Xi_p(-\sqrt{D})^{-1}p^{2s}\right)\sum_{n=0}^{\infty}(\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n))\\
&\quad + (\chi_0\Xi)(\Pi_p)^{-1}p^{s-1/2}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p))\\
&= \left(1+\Xi_p(p)^{-1}p^{2s}\right)p^{-2}\left(\Lambda_p - \Xi_p(\Pi_p)p^{-s+3/2}\right)\left(1-\Lambda_p\Xi_p(\Pi_p)p^{-s-1/2}+\Xi_p(\Pi_p)^2p^{-2s}\right)^{-1}\\
&\quad + \Xi_p(\Pi_p)^{-1}p^{s-1/2}\\
&= \Lambda_p\left(p^{-2}-p^{-1}+\Xi_p(p)^{-1}p^{2s-2}\right)L_p(f,\Xi_p;s)
\end{aligned}$$

from Proposition 5.8. Since $p=2$, we have

$$\begin{aligned}
\mathcal{Z}_p(f,\Xi;s) &= \Lambda_p\left(-p^{-2}+\Xi_p(p)^{-1}p^{2s-2}\right)L_p(f,\Xi_p;s)\\
&= \Lambda_p\Xi_p(p)^{-1}p^{2s-2}\left(1-\Xi_p(p)p^{-2s}\right)L_p(f,\Xi_p;s).
\end{aligned}$$

This completes the proof. □

Lemma 6.9 *Suppose that* $p=2$ *and* $\mathrm{ord}_p D = 3$. *Then we have*

$$\begin{aligned}
\mathcal{Z}_p(f,\Xi;s) &= \left(\Lambda_p\varepsilon_p\chi_{0,p}(-\sqrt{D})\lambda_{K,p}(\psi_p)p^{-1/2}-1\right)\\
&\quad\times \Xi_p(\Pi_p)^{-3}p^{3s-1/2}(1-\Xi_p(p)p^{-2s})L_p(f,\Xi_p;s).
\end{aligned}$$

Proof. We have

$$\mathcal{U}_p = \coprod_{a_p\in\mathbf{Z}_p/p^3\mathbf{Z}_p}\boldsymbol{n}(a_p)S_p\mathcal{U}_0(D)_p \cup \coprod_{a_p\in\mathbf{Z}_p/p^2\mathbf{Z}_p}\overline{\boldsymbol{n}}(pa_p)\mathcal{U}_0(D)_p.$$

This implies that

$$\mathcal{Z}_p(f,\Xi;s) = \mathcal{Z}_p(1)+\mathcal{Z}_p(2)+\mathcal{Z}_p(3)+\mathcal{Z}_p(4)+\mathcal{Z}_p(5),$$

where

$$\begin{aligned}
\mathcal{Z}_p(1) &= \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p))\varphi_{0,p}(y_p) d^\times y_p,\\
\mathcal{Z}_p(2) &= \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p))\overline{\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p))\varphi_{0,p}(y_p)} d^\times y_p,\\
\mathcal{Z}_p(3) &= \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p^2))\overline{\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p^2))\varphi_{0,p}(y_p)} d^\times y_p,\\
\mathcal{Z}_p(4) &= \int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p^2+p))\overline{\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p^2+p))\varphi_{0,p}(y_p)} d^\times y_p,\\
\mathcal{Z}_p(5) &= \sum_{a_p\in\mathbf{Z}_p/p^3\mathbf{Z}_p}\int_{K_p^\times} (\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\boldsymbol{n}(a_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(\boldsymbol{n}(a_p)S_p)\varphi_{0,p}(y_p)} d^\times y_p.
\end{aligned}$$

Since

$$\begin{aligned}
&W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\boldsymbol{n}(a_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(\boldsymbol{n}(a_p)S_p)\varphi_{0,p}(y_p)}\\
&\quad = W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)S_p)\overline{\mathcal{M}^T_{\chi_1}(S_p)\varphi_{0,p}(y_p)}\\
&\quad = W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)S_p)\overline{\lambda_{K,p}(\psi_p)p^{-3/2}\varphi_{0,p}(\sqrt{D}y_p)},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathcal{Z}_p(1)+\mathcal{Z}_p(5)\\
&= \left(1+\varepsilon_p\overline{\lambda_{K,p}(\psi_p)}(\chi_0\Xi)(-\sqrt{D})^{-1}p^{3s}\right)\sum_{n=0}^{\infty}(\chi_0\Xi)(\Pi_p)^n p^{-n(s-1/2)}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^n))\\
&= p^{-1}\left(1+\varepsilon_p\overline{\lambda_{K,p}(\psi_p)}(\chi_0\Xi)(-\sqrt{D})^{-1}p^{3s}\right)\left(C_p(\Lambda_p)-\Xi_p(\Pi_p)p^{-s+1/2}\right)\\
&\quad\times\left(1-\Lambda_p\Xi_p(\Pi_p)p^{-s-1/2}+\Xi_p(\Pi_p)^2p^{-2s}\right)^{-1}
\end{aligned}$$

from Proposition 5.10. Recall that

$$C_p(\Lambda_p)=\Lambda_p-\varepsilon_p e^{-\pi i/4}\omega_p(p)\chi_{0,p}(-\sqrt{D})(1-i\omega_p(1+p))$$

defined in (5.10). Next let $\eta=1$ or $\eta=p+1$. By Lemma 6.6 and Lemma 6.7, we have

$$\begin{aligned}
\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p\eta))\varphi_{0,p}(y_p) &= |D|_p^{-1/2}\, I_p(p\eta,y_p)\\
&= \mathrm{char}_{p^{-1}\mathcal{O}_{K,p}^\times}(y_p)\cdot 2^{-1}\left\{1+\psi_p(p\eta D^{-1}+\kappa(y_p))\right\},
\end{aligned}$$

where $\kappa(y_p)=\mathrm{Tr}(\sqrt{D}^{-1}y_p)$. Hence we obtain

$$\begin{aligned}
&\int_{K_p^\times}(\chi_0\Xi)(y_p)\,|\mathrm{N}(y_p)|_p^{s-1/2}\, W^0_{p,\varepsilon_p}(\boldsymbol{d}(y_p)\overline{\boldsymbol{n}}(p\eta))\overline{\mathcal{M}^T_{\chi_1}(\overline{\boldsymbol{n}}(p\eta))\varphi_{0,p}(y_p)} d^\times y_p\\
&= (\chi_0\Xi)(\Pi_p)^{-2}p^{2s-2}\\
&\quad\times\int_{\mathcal{O}_{K,p}^\times}\left\{1+\overline{\psi_p(p\eta D^{-1}+\kappa(p^{-1}y_p))}\right\}W^0_{p,\varepsilon_p}(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^3\eta\,\mathrm{N}(\Pi_p^2y_p)^{-1}))d^\times y_p.
\end{aligned}$$

Decompose $\mathcal{O}_{K,p}^\times$ as $\mathcal{O}_{K,p}^\times = \mathcal{O}_{K,p}^\times(1) + \mathcal{O}_{K,p}^\times(2)$, where

$$\begin{aligned} \mathcal{O}_{K,p}^\times(1) &= \left\{ z_1 + 2^{-1}\sqrt{D}z_2;\ z_1 \in \mathbf{Z}_p^\times,\ z_2 \in p\mathbf{Z}_p \right\}, \\ \mathcal{O}_{K,p}^\times(2) &= \left\{ z_1 + 2^{-1}\sqrt{D}z_2;\ z_1 \in \mathbf{Z}_p^\times,\ z_2 \in \mathbf{Z}_p^\times \right\}. \end{aligned}$$

Since $\mathrm{N}(\mathcal{O}_{K,p}^\times(1)) \subset 1 + p^3\mathbf{Z}_p$ and $\mathrm{N}(\mathcal{O}_{K,p}^\times(2)) \subset 1 + p + p^2\mathbf{Z}_p$, we have

$$\begin{aligned} &\sum_{\eta=1,\,p+1} \int_{\mathcal{O}_{K,p}^\times} \left\{ 1 + \overline{\psi_p(p\eta D^{-1} + \kappa(p^{-1}y_p))} \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^3\eta\,\mathrm{N}(\Pi_p^2 y_p)^{-1}))d^\times y_p \\ &= \left\{ 1 + \psi_p(-pD^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p)) \int_{\mathcal{O}_{K,p}^\times(1)} d^\times y_p \\ &\quad + \left\{ 1 + \psi_p(-pD^{-1} - p^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p)) \int_{\mathcal{O}_{K,p}^\times(2)} d^\times y_p \\ &\quad + \left\{ 1 + \psi_p(-p(p+1)D^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p)) \int_{\mathcal{O}_{K,p}^\times(1)} d^\times y_p \\ &\quad + \left\{ 1 + \psi_p(-p(p+1)D^{-1} - p^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p)) \int_{\mathcal{O}_{K,p}^\times(2)} d^\times y_p \\ &= \left\{ 1 + \psi_p(-pD^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p)) + \left\{ 1 - \psi_p(-pD^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p)). \end{aligned}$$

Here we used $\psi_p(-p^{-1}) = -1$ and $\psi_p(-p(p+1)D^{-1}) = -\psi_p(-pD^{-1})$. Hence we have

$$\begin{aligned} \mathcal{Z}_p(2) + \mathcal{Z}_p(4) &= (\chi_0\Xi)(\Pi_p)^{-2}p^{2s-2}\Big\{ \left\{ 1 + \psi_p(-pD^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p)) \\ &\qquad + \left\{ 1 - \psi_p(-pD^{-1}) \right\} W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-2})\overline{\boldsymbol{n}}(p^2+p)) \Big\}. \end{aligned}$$

By Lemma 6.6 and Lemma 6.7, we have

$$\mathcal{M}_{\chi_1}^T(\overline{\boldsymbol{n}}(p^2))\varphi_{0,p}(y_p) = |D|_p^{-1/2}\, I_p(p^2, y_p) = \mathrm{char}_{\Pi_p^{-1}\mathcal{O}_{K,p}^\times}(y_p),$$

and get

$$\mathcal{Z}_p(3) = (\chi_0\Xi)(\Pi_p)^{-1}p^{s-1/2}W_{p,\varepsilon_p}^0(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(p^2)).$$

By Proposition 5.10, we obtain

$$\begin{aligned} &\mathcal{Z}_p(2) + \mathcal{Z}_p(3) + \mathcal{Z}_p(4) \\ &= \Xi_p(\Pi_p)^{-1}p^{s-1/2} + \chi_{0,p}(\Pi_p)^{-1}\Xi_p(\Pi_p)^{-2}p^{2s-2}\varepsilon_p B_p \\ &\qquad \times \left\{ 1 + \psi_p(-pD^{-1}) - i\omega_p(1+p)(1 - \psi_p(-pD^{-1})) \right\} \\ &= \Xi_p(\Pi_p)^{-1}p^{s-1/2} \\ &\qquad \times \left\{ 1 + (\chi_0\Xi)(\Pi_p)^{-1}\varepsilon_p B_p p^{s-3/2}\left(1 + \psi_p(-pD^{-1}) - i\omega_p(1+p)(1 - \psi_p(-pD^{-1})) \right) \right\}. \end{aligned}$$

Recall that

$$B_p = e^{-\pi i/4}\chi_{0,p}(-p^{-1}\Pi_p\sqrt{D})$$

defined in (5.9). Therefore we have

$$\mathscr{Z}_p(f,\Xi;s) = \Phi_p(s)L_p(f,\Xi_p;s),$$

where

$$\begin{aligned}
&\Phi_p(s)\\
&= p^{-1}\left(1+\varepsilon_p\overline{\lambda_{K,p}(\psi_p)}(\chi_0\Xi)(-\sqrt{D})^{-1}p^{3s}\right)\left(C_p(\Lambda_p)-\Xi_p(\Pi_p)p^{-s+1/2}\right)\\
&\quad +\Xi_p(\Pi_p)^{-1}p^{s-1/2}\left(1-\Lambda_p\Xi_p(\Pi_p)p^{-s-1/2}+\Xi_p(\Pi_p)^2p^{-2s}\right)\\
&\quad \times\left\{1+(\chi_0\Xi)(\Pi_p)^{-1}\varepsilon_pB_pp^{s-3/2}\left(1+\psi_p(-pD^{-1})-i\omega_p(1+p)(1-\psi_p(-pD^{-1}))\right)\right\}.
\end{aligned}$$

If $p^{-3}D\equiv\pm1 \pmod 4$, then we have $\omega_p(1+p)=\mp1$ and $\omega_p(-1)=\pm1$ respectively. Note that

$$\lambda_{K,p}(\psi_p)=\begin{cases}\omega_p(p) & (p^{-3}D\equiv 1 \pmod 4),\\ -i\omega_p(p) & (p^{-3}D\equiv -1 \pmod 4)\end{cases}$$

by Lemma 3.2, and

$$\psi_p(-pD^{-1})=\begin{cases} i & (p^{-3}D\equiv 1 \pmod 4),\\ -i & (p^{-3}D\equiv -1 \pmod 4).\end{cases}$$

For convenience, put

$$\alpha=\begin{cases} 1 & (p^{-3}D\equiv 1 \pmod 4),\\ -i & (p^{-3}D\equiv -1 \pmod 4)\end{cases}$$

and $X_p=\varepsilon_p\omega_p(p)\chi_{0,p}(-\sqrt{D})$. Then we have $\lambda_{K,p}(\psi_p)=\alpha\omega_p(p)$, $\psi_p(-pD^{-1})=i\alpha^2$, $\omega_p(1+p)=-\alpha^2$ and $\omega_p(-1)=\alpha^2$. Hence we have

$$\begin{aligned}
\Phi_p(s) &= p^{-1}\left(1+\overline{\alpha}X_p^{-1}\Xi_p(\Pi_p)^{-3}p^{3s}\right)\left(\Lambda_p-e^{-\pi i/4}(1+i\alpha^2)X_p-\Xi_p(\Pi_p)p^{-s+1/2}\right)\\
&\quad +\Xi_p(\Pi_p)^{-1}p^{s-1/2}\left(1-\Lambda_p\Xi_p(\Pi_p)p^{-s-1/2}+\Xi_p(\Pi_p)^2p^{-2s}\right)\\
&\quad \times\left\{1+\Xi_p(\Pi_p)^{-1}X_pp^{s-3/2}e^{-\pi i/4}\left(1+i\alpha^2+i\alpha^2(1-i\alpha^2)\right)\right\}.
\end{aligned}$$

We see that $e^{-\pi i/4}(1+i\alpha^2)=\sqrt{p}e^{(\alpha^2-1)\pi i/4}=\sqrt{p}\alpha$, and get

$$\begin{aligned}
\Phi_p(s) &= p^{-1}\left(1+\overline{\alpha}X_p^{-1}\Xi_p(\Pi_p)^{-3}p^{3s}\right)\left(\Lambda_p-\alpha X_pp^{1/2}-\Xi_p(\Pi_p)p^{-s+1/2}\right)\\
&\quad +\Xi_p(\Pi_p)^{-1}p^{s-1/2}\left(1-\Lambda_p\Xi_p(\Pi_p)p^{-s-1/2}+\Xi_p(\Pi_p)^2p^{-2s}\right)\left(1+\alpha X_p\Xi_p(\Pi_p)^{-1}p^s\right)\\
&= \Xi_p(\Pi_p)^{-1}p^{s-1/2}-\Lambda_pX_p\alpha\Xi_p(\Pi_p)^{-1}p^{s-1}+(\alpha X_p-\overline{\alpha}X_p^{-1})\Xi_p(\Pi_p)^{-2}p^{2s-1/2}\\
&\quad -\Xi_p(\Pi_p)^{-3}p^{3s-1/2}+\overline{\alpha}X_p^{-1}\Lambda_p\Xi_p(\Pi_p)^{-3}p^{3s-1}.
\end{aligned}$$

Note that

$$\alpha X_p - \overline{\alpha} X_p^{-1} = \varepsilon_p \omega_p(p) \chi_{0,p}(\sqrt{D})(\alpha^3 - \overline{\alpha}) = 0.$$

Hence we obtain

$$\begin{aligned}
&\Phi_p(s) \\
&= \Xi_p(\Pi_p)^{-1} p^{s-1/2} - \Lambda_p X_p \alpha \Xi_p(\Pi_p)^{-1} p^{s-1} - \Xi_p(\Pi_p)^{-3} p^{3s-1/2} + \Lambda_p X_p \alpha \Xi_p(\Pi_p)^{-3} p^{3s-1} \\
&= \Xi_p(\Pi_p)^{-3} p^{3s-1/2} (1 - \Xi_p(\Pi_p)^2 p^{-2s}) \left(\Lambda_p X_p \alpha p^{-1/2} - 1\right) \\
&= \Xi_p(\Pi_p)^{-3} p^{3s-1/2} (1 - \Xi_p(p) p^{-2s}) \left(\Lambda_p \varepsilon_p \chi_{0,p}(-\sqrt{D}) \lambda_{K,p}(\psi_p) p^{-1/2} - 1\right),
\end{aligned}$$

which completes the proof. □

Finally, we prove our results (Theorem 4.1 and Corollary 4.3).

Proof of Theorem 4.1. Lemma 6.1, 6.2, 6.3, 6.4, 6.5, 6.8 and 6.9 imply

$$\begin{aligned}
\mathcal{Z}(f, \Xi; s) &= \frac{\pi e^{2\pi}}{(4\pi)^{(k+\ell)/2+s-1}} \boldsymbol{W}_{f,2} \Gamma((k+\ell)/2 + s - 1) L(f, \Xi; s) \\
&\quad \times \prod_{p<\infty} (1 - p^{-2s}) \prod_{p|D} \Xi_p(\Pi_p)^{-\operatorname{ord}_p D} D_p(f; s).
\end{aligned}$$

Note that $\Xi_p(p) = 1$ for all $p < \infty$ since $1 = \Xi(p) = \Xi_p(p)$. It is easily seen that $\Xi_p(\Pi_p)^{-\operatorname{ord}_p D} = \Xi_p(\sqrt{D})^{-1}$. Since $\sqrt{D} \in \mathcal{O}_{K,p}^\times$ for $p \nmid D$ and $\Xi_\infty(\sqrt{D}) = (-1)^{(k-\ell)/2}$, we get

$$\begin{aligned}
1 &= \Xi(\sqrt{D}) \\
&= \Xi_\infty(\sqrt{D}) \prod_{p \nmid D} \Xi_p(\sqrt{D}) \prod_{p|D} \Xi_p(\sqrt{D}) \\
&= (-1)^{(k-\ell)/2} \prod_{p|D} \Xi_p(\sqrt{D}).
\end{aligned}$$

Hence we obtain $\prod_{p|D} \Xi_p(\Pi_p)^{-\operatorname{ord}_p D} = (-1)^{(k-\ell)/2}$. Therefore we have

$$\mathcal{Z}(f, \Xi; s) = \frac{(-1)^{(k-\ell)/2} \pi e^{2\pi}}{(4\pi)^{(k+\ell)/2+s-1}} \boldsymbol{W}_{f,2} \Gamma((k+\ell)/2 + s - 1) \zeta(2s)^{-1} L(f, \Xi; s) \prod_{p|D} D_p(f; s).$$

□

Proof of Corollary 4.3. We put $\mathcal{Z}^*(f, \Xi; s) = \pi^{-s} \Gamma(s) \zeta(2s) P_{(k-\ell)/2}(s) \mathcal{Z}(f, \Xi; s)$, where $P_r(s) = \prod_{j=0}^{r} (s + r - j)$. By Proposition 3.3, we have

$$\mathcal{Z}^*(f, \Xi; s) = \mathcal{Z}^*(f, \Xi; 1 - s).$$

Put

$$L^*(f,\Xi;s)=(2\pi)^{-2s}|D|^s\,\Gamma((k-\ell)/2+s+1)\Gamma((k+\ell)/2+s-1)L(f,\Xi;s).$$

Note that $\Gamma(s)P_{(k-\ell)/2}(s)=\Gamma((k-\ell)/2+s+1)$. Therefore Theorem 4.1 implies that

$$\begin{aligned}&|D|^{-s}\prod_{p|D}D_p(f;s)\boldsymbol{W}_{f,2}L^*(f,\Xi;s)\\ =\;&|D|^{s-1}\prod_{p|D}D_p(f;1-s)\boldsymbol{W}_{f,2}L^*(f,\Xi;1-s).\end{aligned}$$

(I) If $\mathrm{ord}_2 D=2$ and $\varepsilon_2=i\chi_{0,2}(\sqrt{D})$, then we obtain

$$\boldsymbol{W}_{f,2}\Lambda_2\{L^*(f,\Xi;s)-L^*(f,\Xi;1-s)\}=0$$

by Theorem 4.1. Therefore we have

$$L^*(f,\Xi;s)=L^*(f,\Xi;1-s)$$

if $\boldsymbol{W}_{f,2}\Lambda_2\neq 0$.

(II) If $\mathrm{ord}_2 D=3$, then we obtain

$$\begin{aligned}\boldsymbol{W}_{f,2}&\left\{\Lambda_2-\sqrt{2}\varepsilon_2\chi_{0,2}(-\sqrt{D})^{-1}\lambda_{K,2}(\psi_2)^{-1}\right\}\\ &\times\{L^*(f,\Xi;s)-L^*(f,\Xi;1-s)\}=0\end{aligned}$$

by Theorem 4.1. Therefore we have

$$L^*(f,\Xi;s)=L^*(f,\Xi;1-s)$$

if $\boldsymbol{W}_{f,2}\left\{\Lambda_2-\sqrt{2}\varepsilon_2\chi_{0,2}(\sqrt{D})\lambda_{K,2}(\psi_2)^{-1}\right\}\neq 0$.

(III) In the remaining case, Theorem 4.1 implies that

$$\boldsymbol{W}_{f,2}\{L^*(f,\Xi;s)-L^*(f,\Xi;1-s)\}=0.$$

Therefore we have

$$L^*(f,\Xi;s)=L^*(f,\Xi;1-s)$$

if $\boldsymbol{W}_{f,2}\neq 0$.

□

7 Classical interpretation

In this section, we state the classical interpretations of cusp forms, Atkin-Lehner operators, Hecke operators and L-function in the case that the class number of K is equal to 1.

7.1 Cusp forms

We put $\Gamma_0(|D|) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z});\, ad-bc=1,\, c \in |D|\,\mathbf{Z} \right\}$. Define a Dirichlet character ω_D by $\omega_D = \prod_{p|D} \omega_p$. Note that $\omega_D(a) = \left(\dfrac{D}{a}\right)$ for $(a, D) = 1$. Let ℓ be a positive even integer. A function F on $\mathfrak{H}$ is called a cusp form on $\Gamma_0(|D|)$ of weight $\ell - 1$ with character ω_D if the following conditions (1) – (3) are satisfied.

(1) F is holomorphic on $\mathfrak{H}$.

(2) For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(|D|)$, we have $F(\gamma z) = \omega_D(d)(cz+d)^{\ell-1}F(z) \quad (z \in \mathfrak{H})$.

(3) $F(z)$ vanishes at each cusp of $\Gamma_0(|D|)$.

We denote by $S_{\ell-1}(\Gamma_0(|D|), \omega_D)$ the space of such functions. We often write z for $h_\infty \langle i \rangle$ $(h_\infty \in H_\infty)$.

Lemma 7.1 *For $f \in S_{\ell-1}(D, \chi_0)$, we put*

$$f_{\mathrm{dm}}(h_\infty \langle i \rangle) = J(h_\infty, i)^{\ell-1} f(h_\infty) \quad (h_\infty \in H_\infty).$$

Then we have $f_{\mathrm{dm}} \in S_{\ell-1}(\Gamma_0(|D|), \omega_D)$.

Proof. The condition (1) is clearly satisfied. Note that $\Gamma_0(|D|) \subset H_{\mathbf{Q}} \cap H_\infty \mathcal{U}_0(D)_f$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(|D|)$, we obtain

$$\begin{aligned} f_{\mathrm{dm}}(\gamma_\infty h_\infty \langle i \rangle) &= J(\gamma_\infty h_\infty, i)^{\ell-1} f(\gamma_\infty h_\infty) \\ &= J(\gamma_\infty, h_\infty \langle i \rangle)^{\ell-1} J(h_\infty, i)^{\ell-1} f(h_\infty \gamma_f^{-1}) \quad (\gamma_f \in H_{\mathbf{A},f}) \\ &= \prod_{p|D} \chi_{0,p}(d)(cz+d)^{\ell-1} J(h_\infty, i)^{\ell-1} f(h_\infty) \\ &= \omega_D(d)(cz+d)^{\ell-1} f_{\mathrm{dm}}(h_\infty \langle i \rangle). \end{aligned}$$

Hence the condition (2) is satisfied. The function f_{dm} has a period 1. Then we have the Fourier expansion

$$f_{\mathrm{dm}}(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z},$$

where

$$c(n) = \int_0^1 f_{\mathrm{dm}}(z+u) e^{-2\pi i n (z+u)} du.$$

Recall that

$$\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} f(\boldsymbol{n}(x)h)dx = 0 \quad (h \in H_{\mathbf{A}})$$

for $f \in S_{\ell-1}(D, \chi_0)$. Since

$$\begin{aligned}\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} f(\boldsymbol{n}(x)h_\infty)dx &= \int_0^1 dx_\infty \int_{\mathbf{Z}_f} dx_f \, f(\boldsymbol{n}(x_\infty)\boldsymbol{n}(x_f)h_\infty) \\ &= \int_0^1 f(\boldsymbol{n}(x_\infty)h_\infty)dx_\infty \\ &= J(h_\infty, i)^{1-\ell} \int_0^1 f_{\mathrm{dm}}(z + x_\infty)dx_\infty \\ &= J(h_\infty, i)^{1-\ell} c(0),\end{aligned}$$

we obtain $c(0) = 0$. From this, we have $f_{\mathrm{dm}}(i\infty) = 0$. For a cusp $x \neq i\infty$, we can prove that $f_{\mathrm{dm}}(x) = 0$ in a similar way. This completes the proof. □

7.2 Atkin-Lehner operators

For $f \in S_{\ell-1}(D, \chi_0)$, we put

$$(Wf_{\mathrm{dm}})(z) = (\sqrt{-D}z)^{1-\ell} f_{\mathrm{dm}}\left(\frac{1}{Dz}\right).$$

Then $Wf_{\mathrm{dm}} \in S_{\ell-1}(\Gamma_0(|D|), \omega_D)$.

Lemma 7.2 *For $f \in S_{\ell-1}(D, \chi_0)$, put $\mathfrak{F}_f = (\prod_{p|D} \mathfrak{F}_{D,p}) f$. Namely $\mathfrak{F}_f(h) = f(h \prod_{p|D} w_{D,p})$. Let $\mathfrak{F}_{f,\mathrm{dm}}(z) = J(h_\infty, i)^{\ell-1} \mathfrak{F}_f(h_\infty)$. Then we have*

$$\mathfrak{F}_{f,\mathrm{dm}} = -i^{\ell-2} \prod_{p|D} \chi_{0,p}(\sqrt{D})^{-1} (Wf_{\mathrm{dm}}).$$

Proof. Since $w_{D,p} \in \mathcal{U}_p$ and $\sqrt{D} \in \mathcal{O}_{K,p}^{\times}$ for $p \nmid D$, we have

$$\begin{aligned}
\mathfrak{F}_{f,\mathrm{dm}}(z) &= J(h_\infty, i)^{\ell-1} f(h_\infty \prod_{p|D} w_{D,p}) \\
&= J(h_\infty, i)^{\ell-1} f(w_{D,p,\infty}^{-1} h_\infty \prod_{p\nmid D} w_{D,p}^{-1}) \\
&= J(w_{D,p,\infty}^{-1}, h_\infty \langle i \rangle)^{1-\ell} J(w_{D,p,\infty}^{-1} h_\infty, i)^{\ell-1} \prod_{p\nmid D} \chi_{0,p}(\sqrt{D}) f(w_{D,p,\infty}^{-1} h_\infty) \\
&= \chi_{0,\infty}(\sqrt{D})^{-1} \prod_{p|D} \chi_{0,p}(\sqrt{D})^{-1} J(w_{D,p,\infty}^{-1}, z)^{1-\ell} f_{\mathrm{dm}}(w_{D,p,\infty}^{-1} z) \\
&= (\sqrt{D}/\sqrt{-D}) \prod_{p|D} \chi_{0,p}(\sqrt{D})^{-1} (-\sqrt{D}z)^{1-\ell} f_{\mathrm{dm}}\left(\frac{1}{Dz}\right) \\
&= -\prod_{p|D} \chi_{0,p}(\sqrt{D})^{-1} (-\sqrt{D}/\sqrt{-D})^{2-\ell} (W f_{\mathrm{dm}})(z) \\
&= -i^{\ell-2} \prod_{p|D} \chi_{0,p}(\sqrt{D})^{-1} (W f_{\mathrm{dm}})(z).
\end{aligned}$$

□

7.3 Hecke operators

For $F \in S_{\ell-1}(\Gamma_0(|D|), \omega_D)$ and a positive integer n, we define the (classical) Hecke operator T_n by

$$T_n F(z) = n^{\ell-2} \sum_{ad=n} \sum_{b=0}^{d-1} \omega_D(a) d^{-\ell+1} F\left(\frac{az+b}{d}\right).$$

Here we make a convention that $\omega_D(a) = 0$ if $(a, D) \neq 1$. We use the following facts in later discussion.

- If p is inert in $K/\mathbf{Q}$,

$$T_{p^2} F(z) = p^{-2} \sum_{b=0}^{p^2-1} F\left(\frac{z+b}{p^2}\right) - p^{\ell-3} \sum_{b=0}^{p-1} F\left(z + \frac{b}{p}\right) + p^{2\ell-4} F(p^2 z).$$

- If p splits in $K/\mathbf{Q}$,

$$T_p F(z) = p^{-1} \sum_{b=0}^{p-1} F\left(\frac{z+b}{p}\right) + p^{\ell-2} F(pz).$$

- If p ramifies in $K/\mathbf{Q}$,

$$T_pF(z) = p^{-1}\sum_{b=0}^{p-1} F\left(\frac{z+b}{p}\right).$$

In this case, we also have

$$WT_pWF(z) = -p^{\ell-2}\sum_{b=0}^{p-1}(Dbz+1)^{1-\ell}F\left(\frac{pz}{Dbz+1}\right).$$

Recall that $WF(z) = (\sqrt{-D}z)^{1-\ell}F\left(\frac{1}{Dz}\right)$.

Lemma 7.3 *For $f \in S_{\ell-1}(D,\chi_0)$, we have the following.*

(1) *If p is inert in $K/\mathbf{Q}$,*

$$(\mathcal{T}_pf)_{\mathrm{dm}}(z) = p^{-\ell+3}T_{p^2}f_{\mathrm{dm}}(z) + f_{\mathrm{dm}}(z).$$

(2) *If p splits in $K/\mathbf{Q}$,*

$$(\mathcal{T}_{p,j}f)_{\mathrm{dm}}(z) = p^{3/2}\Pi_{p,j}^{-\ell}T_pf_{\mathrm{dm}}(z) \quad (j=1,2)$$

(3) *If p ramifies in $K/\mathbf{Q}$,*

$$(\mathcal{T}_pf)_{\mathrm{dm}}(z) = p^{3/2}(\Pi_p^{\sigma})^{-\ell}T_pf_{\mathrm{dm}}(z) - p^{3/2}\Pi_p^{-\ell}(WT_pWf_{\mathrm{dm}})(z).$$

Proof. We first suppose that p is inert in $K/\mathbf{Q}$. Note that $\omega_p(p) = -1$ in this case. We have

$$\begin{aligned}
&(\mathcal{T}_p f)_{\mathrm{dm}}(z) \\
&= J(h_\infty, i)^{\ell-1}(\mathcal{T}_p f)(h_\infty) \\
&= -J(h_\infty, i)^{\ell-1} f(h_\infty \boldsymbol{d}(p^{-1})) - J(h_\infty, i)^{\ell-1} \sum_{x \in \mathbf{Z}_p^\times / p\mathbf{Z}_p} f(h_\infty \boldsymbol{n}(p^{-1}x)) \\
&\quad - J(h_\infty, i)^{\ell-1} \sum_{y \in \mathbf{Z}_p / p^2 \mathbf{Z}_p} f(h_\infty \boldsymbol{n}(y) \boldsymbol{d}(p)) \\
&= -J(h_\infty, i)^{\ell-1} \omega_p(p) f(\boldsymbol{d}(p)_\infty h_\infty) - J(h_\infty, i)^{\ell-1} \sum_{x=1}^{p-1} f(\boldsymbol{n}(-p^{-1}x)_\infty h_\infty) \\
&\quad - J(h_\infty, i)^{\ell-1} \omega_p(p) \sum_{y=0}^{p^2-1} f(\boldsymbol{d}(p^{-1})_\infty \boldsymbol{n}(-y)_\infty h_\infty) \\
&= p^{\ell-1} f_{\mathrm{dm}}(\boldsymbol{d}(p)_\infty h_\infty \langle i \rangle) - \sum_{x=1}^{p-1} f_{\mathrm{dm}}(\boldsymbol{n}(-p^{-1}x)_\infty h_\infty \langle i \rangle) \\
&\quad + p^{1-\ell} \sum_{y=0}^{p^2-1} f_{\mathrm{dm}}(\boldsymbol{d}(p^{-1})_\infty \boldsymbol{n}(-y)_\infty h_\infty \langle i \rangle) \\
&= p^{\ell-1} f_{\mathrm{dm}}(p^2 z) - \sum_{x=1}^{p-1} f_{\mathrm{dm}}\left(z + \frac{x}{p}\right) + p^{1-\ell} \sum_{y=0}^{p^2-1} f_{\mathrm{dm}}\left(\frac{z+y}{p^2}\right) \\
&= p^{\ell-1} f_{\mathrm{dm}}(p^2 z) - \sum_{x=0}^{p-1} f_{\mathrm{dm}}\left(z + \frac{x}{p}\right) + f_{\mathrm{dm}}(z) + p^{1-\ell} \sum_{y=0}^{p^2-1} f_{\mathrm{dm}}\left(\frac{z+y}{p^2}\right) \\
&= p^{-\ell+3} T_{p^2} f_{\mathrm{dm}}(z) + f_{\mathrm{dm}}(z).
\end{aligned}$$

Next suppose that p splits in $K/\mathbf{Q}$. Note that $\omega_p(p)=1$. For $j=1,2$, we have

$$\begin{aligned}
&(\mathcal{T}_{p,j}f)_{\mathrm{dm}}(z)\\
&= J(h_\infty,i)^{\ell-1}(\mathcal{T}_{p,j}f)(h_\infty)\\
&= J(h_\infty,i)^{\ell-1}\chi_{0,p}(\Pi_{p,j})^{-1}\left\{f(h_\infty\boldsymbol{d}(\Pi_{p,j}^{-1}))+\sum_{x\in\mathbf{Z}_p/p\mathbf{Z}_p}f(h_\infty\boldsymbol{n}(x)\boldsymbol{d}(\Pi_{p,j}^{\sigma}))\right\}\\
&= J(h_\infty,i)^{\ell-1}\chi_{0,p}(\Pi_{p,j})^{-1}f(\boldsymbol{d}(\Pi_{p,j})_\infty h_\infty)\prod_{q\neq p}\chi_{0,q}(\Pi_{p,j}^{\sigma})\\
&\quad + J(h_\infty,i)^{\ell-1}\chi_{0,p}(\Pi_{p,j})^{-1}\sum_{x=0}^{p-1}f(\boldsymbol{d}((\Pi_{p,j}^{\sigma})^{-1})_\infty\boldsymbol{n}(-x)_\infty h_\infty)\prod_{q\neq p}\chi_{0,q}(\Pi_{p,j}^{-1})\\
&= (\Pi_{p,j}^{\sigma})^{\ell-1}\chi_{0,\infty}(\Pi_{p,j}^{\sigma})^{-1}f_{\mathrm{dm}}(\boldsymbol{d}(\Pi_{p,j})_\infty h_\infty\langle i\rangle)\\
&\quad + \Pi_{p,j}^{1-\ell}\chi_{0,\infty}(\Pi_{p,j})\sum_{x=0}^{p-1}f_{\mathrm{dm}}(\boldsymbol{d}((\Pi_{p,j}^{\sigma})^{-1})_\infty\boldsymbol{n}(-x)_\infty h_\infty\langle i\rangle)\\
&= \sqrt{p}^{-1}(\Pi_{p,j}^{\sigma})^{\ell}f_{\mathrm{dm}}(pz)+\sqrt{p}\,\Pi_{p,j}^{-\ell}\sum_{x=0}^{p-1}f_{\mathrm{dm}}\left(\frac{z+x}{p}\right)\\
&= p^{3/2}\Pi_{p,j}^{-\ell}T_pf_{\mathrm{dm}}(z).
\end{aligned}$$

Finally, we suppose that p ramifies in $K/\mathbf{Q}$. Put $\mathcal{T}_pf=\mathcal{T}_{p,+}f+\mathcal{T}_{p,-}f$, where

$$\begin{aligned}
\mathcal{T}_{p,+}f(h) &= \chi_{0,p}(\Pi_p)\sum_{x\in\mathbf{Z}_p/p\mathbf{Z}_p}f(h\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p)),\\
\mathcal{T}_{p,-}f(h) &= \chi_{0,p}(\Pi_p)^{-1}\sum_{y\in\mathbf{Z}_p/p\mathbf{Z}_p}f(h\overline{\boldsymbol{n}}(Dy)\boldsymbol{d}(\Pi_p^{-1})).
\end{aligned}$$

Then we have

$$\begin{aligned}
(\mathcal{T}_pf)_{\mathrm{dm}}(z) &= J(h_\infty,i)^{\ell-1}(\mathcal{T}_pf)(h_\infty)\\
&= J(h_\infty,i)^{\ell-1}(\mathcal{T}_{p,+}f)(h_\infty)+J(h_\infty,i)^{\ell-1}(\mathcal{T}_{p,-}f)(h_\infty).
\end{aligned}$$

First we have

$$
\begin{aligned}
&J(h_\infty, i)^{\ell-1}(\mathcal{T}_{p,+}f)(h_\infty)\\
&= J(h_\infty, i)^{\ell-1}\chi_{0,p}(\Pi_p)\sum_{x\in\mathbf{Z}_p/p\mathbf{Z}_p} f(h_\infty \boldsymbol{n}(x)\boldsymbol{d}(\Pi_p))\\
&= \chi_{0,p}(\Pi_p)\sum_{x=0}^{p-1} J(\boldsymbol{d}(\Pi_p^{-1})_\infty \boldsymbol{n}(-x)_\infty, z)^{1-\ell} J(\boldsymbol{d}(\Pi_p^{-1})_\infty \boldsymbol{n}(-x)_\infty h_\infty, i)^{\ell-1}\\
&\quad f(\boldsymbol{d}(\Pi_p^{-1})_\infty \boldsymbol{n}(-x)_\infty h_\infty)\prod_{q\neq p}\chi_{0,q}(\Pi_p^\sigma)^{-1}\\
&= \chi_{0,p}(\Pi_p)\chi_{0,p}(\Pi_p^\sigma)\chi_{0,\infty}(\Pi_p^\sigma)(\Pi_p^\sigma)^{1-\ell}\sum_{x=0}^{p-1} f_{\mathrm{dm}}(\boldsymbol{d}(\Pi_p^{-1})_\infty \boldsymbol{n}(-x)_\infty h_\infty \langle i\rangle)\\
&= \sqrt{p}(\Pi_p^\sigma)^{-\ell}\sum_{x=0}^{p-1} f_{\mathrm{dm}}\left(\frac{z+x}{p}\right)\\
&= p^{3/2}(\Pi_p^\sigma)^{-\ell} T_p f_{\mathrm{dm}}(z).
\end{aligned}
$$

We next have

$$
\begin{aligned}
&J(h_\infty, i)^{\ell-1}(\mathcal{T}_{p,-}f)(h_\infty)\\
&= J(h_\infty, i)^{\ell-1}\chi_{0,p}(\Pi_p)^{-1}\sum_{y\in\mathbf{Z}_p/p\mathbf{Z}_p} f(h_\infty \overline{\boldsymbol{n}}(Dy)\boldsymbol{d}(\Pi_p^{-1}))\\
&= \chi_{0,p}(\Pi_p)^{-1}\sum_{y=0}^{p-1} J(\boldsymbol{d}(\Pi_p)_\infty \overline{\boldsymbol{n}}(-Dy)_\infty, z)^{1-\ell} J(\boldsymbol{d}(\Pi_p)_\infty \overline{\boldsymbol{n}}(-Dy)_\infty h_\infty, i)^{\ell-1}\\
&\quad f(\boldsymbol{d}(\Pi_p)_\infty \overline{\boldsymbol{n}}(-Dy)_\infty h_\infty)\prod_{q\neq p}\chi_{0,q}(\Pi_p^\sigma)\\
&= \chi_{0,p}(\Pi_p)^{-1}\chi_{0,p}(\Pi_p^\sigma)^{-1}\chi_{0,\infty}(\Pi_p^\sigma)^{-1}(\Pi_p^\sigma)^{\ell-1}\\
&\quad \sum_{y=0}^{p-1}(-Dyz+1)^{1-\ell} f_{\mathrm{dm}}(\boldsymbol{d}(\Pi_p)_\infty \overline{\boldsymbol{n}}(-Dy)_\infty h_\infty \langle i\rangle)\\
&= \sqrt{p}^{-1}(\Pi_p^\sigma)^{\ell}\sum_{y=0}^{p-1}(Dyz+1)^{1-\ell} f_{\mathrm{dm}}\left(\frac{pz}{Dyz+1}\right)\\
&= -p^{3/2}\Pi_p^{-\ell}(WT_pWf_{\mathrm{dm}})(z).
\end{aligned}
$$

Hence we obtain

$$
(\mathcal{T}_p f)_{\mathrm{dm}}(z) = p^{3/2}(\Pi_p^\sigma)^{-\ell} T_p f_{\mathrm{dm}}(z) - p^{3/2}\Pi_p^{-\ell}(WT_pWf_{\mathrm{dm}})(z).
$$

□

7.4 L-function

For $F(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \in S_{\ell-1}(\Gamma_0(|D|), \omega_D)$, put

$$KF(z) = \overline{F(-\overline{z})} = \sum_{n=1}^{\infty} \overline{c(n)} e^{2\pi i n z}.$$

Then we have

$$KKF(z) = F(z),$$

and

$$WK = -KW$$

since

$$\begin{aligned} KWF(z) &= \overline{\left\{\sqrt{-D}(-\overline{z})\right\}^{1-\ell} F\left(-\frac{1}{D\overline{z}}\right)} \\ &= (-1)^{1-\ell}(\sqrt{-D}z)^{1-\ell}\overline{F\left(-\frac{1}{D\overline{z}}\right)} \\ &= -(\sqrt{-D}z)^{1-\ell} KF\left(\frac{1}{Dz}\right) \\ &= -WKF(z). \end{aligned}$$

We assume that $F(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}$ is a nonzero form in $S_{\ell-1}(\Gamma_0(|D|), \omega_D)$. We call F a normalized newform (in the sense of [Li]) if the following conditions hold.

(1) $T_p F = \lambda_p F$ for $p \nmid D$ ($\lambda_p \in \mathbf{C}$).

(2) $c(1) = 1$.

Lemma 7.4 ([Li]) *Let $F(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \in S_{\ell-1}(\Gamma_0(|D|), \omega_D)$ be a normalized newform.*

(i) *For any prime p, we have $T_p F = c(p) F$ and*

$$c(p^m)c(p^n) = \sum_{j=0}^{\min(m,n)} \omega(p^j) p^{(\ell-2)j} c(p^{m+n-2j}) \quad (m, n \geq 0).$$

(ii) *We have* $|c(p)| = p^{(\ell-2)/2}$ *for* $p \mid D$.

(iii) *We have* $KWF = \gamma F \quad (\gamma = \pm 1)$.

For $f \in S_{\ell-1}(D, \chi_0)$ and $s \in \mathbf{C}$, define a Rankin L-function

$$Z(f_{\mathrm{dm}}; s) = \sum_{\mathfrak{a}} c(\mathrm{N}\,\mathfrak{a})\alpha^{\ell}\,\mathrm{N}\,\mathfrak{a}^{-s},$$

where $\mathfrak{a} = \alpha\mathcal{O}_K$ runs over the nonzero integral ideals of K (cf. [Ran], [Sel]). The object of this subsection is to show the following result.

Proposition 7.5 *Let* $f \in S_{\ell-1}(D, \chi_0)$ *and assume that* f_{dm} *is a normalized newform. Then we have*

$$L(f, \mathbf{1}; s) = \zeta(2s) Z(f_{\mathrm{dm}}; s + \ell - 1) \prod_{p \mid D} (1 - p^{-2s}) \left(1 - \overline{c(p)\Pi_p^{\ell}} p^{-s-\ell+1}\right)^{-1}.$$

Lemma 7.6

(1) *If* p *is inert in* $K/\mathbf{Q}$,

$$L_p(f, \mathbf{1}; s) = \left\{1 - (p^{-\ell+2}c(p^2) + 1)p^{-2s} + p^{-4s}\right\}^{-1}.$$

(2) *If* p *splits in* $K/\mathbf{Q}$,

$$L_p(f, \mathbf{1}; s) = \prod_{j=1,2} \left\{1 - \Pi_{p,j}^{-\ell} c(p) p^{-s+1} + (\Pi_{p,j}/\Pi_{p,j}^{\sigma})^{-\ell} p^{-2s}\right\}^{-1}.$$

(3) *If* p *ramifies in* $K/\mathbf{Q}$,

$$L_p(f, \mathbf{1}; s) = \left(1 - (\Pi_p^{\sigma})^{-\ell} c(p) p^{-s+1}\right)^{-1} \left(1 - \Pi_p^{-\ell}\overline{c(p)} p^{-s+1}\right)^{-1}.$$

Proof. First suppose that p is inert in $K/\mathbf{Q}$. Since

$$\begin{aligned} T_p(T_p f)(z) &= p^{2(\ell-2)} f(p^2 z) - p^{\ell-3} \sum_{b=0}^{p-1} f\left(z + \frac{b}{p}\right) \\ &\quad - p^{\ell-3} \sum_{b=0}^{p-1} f(z+b) + p^{-2} \sum_{b=0}^{p-1} \sum_{B=0}^{p-1} f\left(\frac{z+b+pB}{p^2}\right) \\ &= p^{2(\ell-2)} f(p^2 z) - p^{\ell-2} f(z) \\ &\quad - p^{\ell-3} \sum_{b=0}^{p-1} f\left(z + \frac{b}{p}\right) + p^{-2} \sum_{b=0}^{p^2-1} f\left(\frac{z+b}{p^2}\right), \end{aligned}$$

we have

$$\begin{aligned} T_{p^2}f(z) &= p^{2(\ell-2)}f(p^2z) - p^{\ell-3}\sum_{b=0}^{p-1} f\left(z+\frac{b}{p}\right) + p^{-2}\sum_{b=0}^{p^2-1} f\left(\frac{z+b}{p^2}\right) \\ &= T_p(T_pf)(z) + p^{\ell-2}f(z). \end{aligned}$$

Hence we obtain

$$T_{p^2}f(z) = \left\{c(p)^2 + p^{\ell-2}\right\} f(z) = c(p^2)f(z)$$

if $T_pf(z) = c(p)f(z)$. Here we used $c(p)^2 = c(p^2) - p^{\ell-2}$. Then

$$\Lambda_p = c(p^2)p^{-\ell+3} + 1.$$

Therefore we have

$$\begin{aligned} L_p(f,\mathbf{1};s) &= \left\{1 + (1-p-\Lambda_p)p^{-2s-1} + p^{-4s}\right\}^{-1} \\ &= \left\{1 + (-c(p^2)p^{-\ell+3} - p)p^{-2s-1} + p^{-4s}\right\}^{-1} \\ &= \left\{1 - (p^{-\ell+2}c(p^2) + 1)p^{-2s} + p^{-4s}\right\}^{-1}. \end{aligned}$$

We next suppose that p splits in $K/\mathbf{Q}$. It is easily seen that

$$\Lambda_{p,j} = p^{3/2}\Pi_{p,j}^{-\ell}c(p) \quad (j=1,2).$$

Hence we have

$$\begin{aligned} L_p(f,\mathbf{1};s) &= \prod_{j=1,2}\left\{1 - \Lambda_{p,j}p^{-s-1/2} + \Omega(\Pi_{p,j}/\Pi_{p,j}^{\sigma})p^{-2s}\right\}^{-1} \\ &= \prod_{j=1,2}\left\{1 - \Pi_{p,j}^{-\ell}c(p)p^{-s+1} + (\Pi_{p,j}^{\sigma}/\Pi_{p,j})^{\ell}p^{-2s}\right\}^{-1} \\ &= \prod_{j=1,2}\left\{1 - \Pi_{p,j}^{-\ell}c(p)p^{-s+1} + (\Pi_{p,j}/\Pi_{p,j}^{\sigma})^{-\ell}p^{-2s}\right\}^{-1}. \end{aligned}$$

Finally we suppose that p ramifies in $K/\mathbf{Q}$. Since

$$Wf_{\mathrm{dm}}(z) = KKWf_{\mathrm{dm}}(z) = \gamma Kf_{\mathrm{dm}}(z),$$

we have

$$T_pWf_{\mathrm{dm}}(z) = \gamma T_pKf_{\mathrm{dm}}(z) = \gamma\overline{c(p)}Kf_{\mathrm{dm}}(z)$$

and

$$\begin{aligned} WT_pWf_{\mathrm{dm}}(z) &= W(T_pWf_{\mathrm{dm}})(z) \\ &= \gamma\overline{c(p)}WKf_{\mathrm{dm}}(z) \\ &= -\gamma\overline{c(p)}KWf_{\mathrm{dm}}(z) \\ &= -\overline{c(p)}f_{\mathrm{dm}}(z). \end{aligned}$$

Recall that

$$(\mathcal{T}_p f)_{\mathrm{dm}} = p^{3/2}(\Pi_p^\sigma)^{-\ell} T_p f_{\mathrm{dm}} - p^{3/2}\Pi_p^{-\ell}(WT_pWf_{\mathrm{dm}}).$$

Hence

$$\Lambda_p = p^{3/2}(\Pi_p^\sigma)^{-\ell} c(p) + p^{3/2}\Pi_p^{-\ell}\overline{c(p)}.$$

Therefore we obtain

$$\begin{aligned} L_p(f,\mathbf{1};s) &= \left(1 - \Lambda_p p^{-s-1/2} + p^{-2s}\right)^{-1} \\ &= \left(1 - p^{-s+1}(\Pi_p^\sigma)^{-\ell} c(p) - p^{-s+1}\Pi_p^{-\ell}\overline{c(p)} + p^{-2s}\right)^{-1} \\ &= \left(1 - (\Pi_p^\sigma)^{-\ell} c(p) p^{-s+1}\right)^{-1}\left(1 - \Pi_p^{-\ell}\overline{c(p)} p^{-s+1}\right)^{-1}. \end{aligned}$$

□

Let $Z(f_{\mathrm{dm}};s) = \prod_{p<\infty} Z_p(f_{\mathrm{dm}};s)$, where

$$Z_p(f_{\mathrm{dm}};s) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} c(p^{2k}) p^{(\ell-2s)k} & (p \text{ is inert in } K/\mathbf{Q}), \\ \displaystyle\sum_{k_1,k_2=0}^{\infty} c(p^{k_1+k_2}) \Pi_{p,1}^{\ell k_1}\Pi_{p,2}^{\ell k_2} p^{-(k_1+k_2)s} & (p \text{ splits in } K/\mathbf{Q}), \\ \displaystyle\sum_{k=0}^{\infty} c(p^k)\Pi_p^{\ell k} p^{-ks} & (p \text{ ramifies in } K/\mathbf{Q}). \end{cases}$$

Lemma 7.7

(1) *If p is inert in $K/\mathbf{Q}$,*

$$Z_p(f_{\mathrm{dm}};s) = (1 - p^{2\ell-2s-2}) L_p(f,\mathbf{1};s-\ell+1).$$

(2) *If p splits in $K/\mathbf{Q}$,*

$$Z_p(f_{\mathrm{dm}};s) = (1 - p^{2\ell-2s-2}) L_p(f,\mathbf{1};s-\ell+1).$$

(3) *If p ramifies in $K/\mathbf{Q}$,*

$$Z_p(f_{\mathrm{dm}};s) = \left(1 - \overline{c(p)}(\Pi_p^\sigma)^{\ell} p^{-s}\right) L_p(f,\mathbf{1};s-\ell+1).$$

Proof. We first suppose that p is inert in $K/\mathbf{Q}$. We write $Z_p(f_{\mathrm{dm}};s)=\varphi(p^{\ell-2s})$, where

$$\varphi(X)=\sum_{k=0}^{\infty}c(p^{2k})X^k.$$

Since

$$c(p^2)c(p^{2k})=c(p^{2k+2})-p^{\ell-2}c(p^{2k})+p^{2(\ell-2)}c(p^{2k-2})\quad (k\geq 1),$$

we obtain

$$X^{-1}\left\{\varphi(X)-c(p^2)X-1\right\}-\left\{c(p^2)+p^{\ell-2}\right\}\left\{\varphi(X)-1\right\}+p^{2(\ell-2)}X\varphi(X)=0.$$

This equation implies that

$$\varphi(X)=\frac{1-p^{\ell-2}X}{1-\left\{c(p^2)+p^{\ell-2}\right\}X+p^{2(\ell-2)}X^2}.$$

Hence we have

$$\begin{aligned}Z_p(f_{\mathrm{dm}};s)&=\varphi(p^{\ell-2s})\\&=\frac{1-p^{2\ell-2s-2}}{1-(p^{-\ell+2}c(p^2)+1)p^{2\ell-2s-2}+p^{4\ell-4s-4}}\\&=(1-p^{2\ell-2s-2})L_p(f,\mathbf{1};s-\ell+1).\end{aligned}$$

Next suppose that p splits in $K/\mathbf{Q}$. Then we get

$$\begin{aligned}Z_p(f_{\mathrm{dm}};s)&=\sum_{k=0}^{\infty}c(p^k)p^{-ks}\sum_{r=0}^{k}(\Pi_{p,1}^{\ell})^{k-r}(\Pi_{p,2}^{\ell})^{r}\\&=\sum_{k=0}^{\infty}c(p^k)p^{-ks}\frac{(\Pi_{p,1}^{\ell})^{k+1}-(\Pi_{p,2}^{\ell})^{k+1}}{\Pi_{p,1}^{\ell}-\Pi_{p,2}^{\ell}}\\&=\frac{1}{\Pi_{p,1}^{\ell}-\Pi_{p,2}^{\ell}}\left\{\Pi_{p,1}^{\ell}\sum_{k=0}^{\infty}c(p^k)(\Pi_{p,1}^{\ell}p^{-s})^k-\Pi_{p,2}^{\ell}\sum_{k=0}^{\infty}c(p^k)(\Pi_{p,2}^{\ell}p^{-s})^k\right\}\\&=\frac{1}{\Pi_{p,1}^{\ell}-\Pi_{p,2}^{\ell}}\left\{\Pi_{p,1}^{\ell}\varphi(\Pi_{p,1}^{\ell}p^{-s})-\Pi_{p,2}^{\ell}\varphi(\Pi_{p,2}^{\ell}p^{-s})\right\},\end{aligned}$$

where

$$\varphi(X)=\sum_{k=0}^{\infty}c(p^k)X^k.$$

Since

$$c(p^{k+1})-c(p)c(p^k)+p^{\ell-2}c(p^{k-1})=0\quad (k\geq 1),$$

we obtain

$$X^{-1}\{\varphi(X)-c(p)X-1\}-c(p)\{\varphi(X)-1\}+p^{\ell-2}X\varphi(X)=0.$$

From this, we have

$$\varphi(X)=\frac{1}{1-c(p)X+p^{\ell-2}X^2}.$$

Hence

$$\begin{aligned} Z_p(f_{\mathrm{dm}};s) &= \frac{1}{\Pi_{p,1}^{\ell}-\Pi_{p,2}^{\ell}}\left\{\Pi_{p,1}^{\ell}\varphi(\Pi_{p,1}^{\ell}p^{-s})-\Pi_{p,2}^{\ell}\varphi(\Pi_{p,2}^{\ell}p^{-s})\right\} \\ &= (1-p^{2\ell-2s-2})\prod_{j=1,2}\left\{1-c(p)\Pi_{p,j}^{\ell}p^{-s}+\Pi_{p,j}^{2\ell}p^{\ell-2s-2}\right\}^{-1}. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} L_p(f,\mathbf{1};s-\ell+1) &= \prod_{j=1,2}\left\{1-\Pi_{p,j}^{-\ell}c(p)p^{-s+\ell}+(\Pi_{p,j}/\Pi_{p,j}^{\sigma})^{-\ell}p^{-2s+2\ell-2}\right\}^{-1} \\ &= \prod_{j=1,2}\left\{1-c(p)(\Pi_{p,j}^{\sigma})^{\ell}p^{-s}+(\Pi_{p,j}^{\sigma})^{2\ell}p^{\ell-2s-2}\right\}^{-1}. \end{aligned}$$

Therefore we have

$$Z_p(f_{\mathrm{dm}};s)=(1-p^{2\ell-2s-2})L_p(f,\mathbf{1};s-\ell+1).$$

Finally suppose that p ramifies in $K/\mathbf{Q}$. We write $Z_p(f_{\mathrm{dm}};s)=\varphi(\Pi_p^{\ell}p^{-s})$, where

$$\varphi(X)=\sum_{k=0}^{\infty}c(p^k)X^k.$$

Since

$$c(p)c(p^k)=c(p^{k+1})\quad(k\geq 0),$$

we have

$$c(p)\varphi(X)-X^{-1}\{\varphi(X)-1\}=0.$$

Hence we get

$$\varphi(X)=(1-c(p)X)^{-1}.$$

On the other hand, we obtain

$$\begin{aligned} L_p(f,\mathbf{1};s-\ell+1) &= \left(1-(\Pi_p^{\sigma})^{-\ell}c(p)p^{-s+\ell}\right)^{-1}\left(1-\Pi_p^{-\ell}\overline{c(p)}p^{-s+\ell}\right)^{-1} \\ &= \left(1-\Pi_p^{\ell}c(p)p^{-s}\right)^{-1}\left(1-(\Pi_p^{\sigma})^{\ell}\overline{c(p)}p^{-s}\right)^{-1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} Z_p(f_{\mathrm{dm}};s) &= \varphi(\Pi_p^{\ell} p^{-s}) \\ &= \left(1-\Pi_p^{\ell} c(p) p^{-s}\right)^{-1} \\ &= \left(1-\overline{c(p)}(\Pi_p^{\sigma})^{\ell} p^{-s}\right) L_p(f,\mathbf{1};s-\ell+1). \end{aligned}$$

This completes the proof. □

Proof of Proposition 7.5. By Lemma 7.6 and 7.7, we obtain

$$\begin{aligned} & Z(f_{\mathrm{dm}};s) \\ &= \prod_{p|D}\left(1-\overline{c(p)}(\Pi_p^{\sigma})^{\ell}p^{-s}\right)\prod_{p\nmid D}\left(1-p^{-2s+2\ell-2}\right) L(f,\mathbf{1};s-\ell+1) \\ &= \zeta(2s-2\ell+2)^{-1}L(f,\mathbf{1};s-\ell+1)\prod_{p|D}(1-p^{-2s+2\ell-2})^{-1}\left(1-\overline{c(p)}(\Pi_p^{\sigma})^{\ell}p^{-s}\right). \end{aligned}$$

Therefore we have

$$L(f,\mathbf{1};s)=\zeta(2s)Z(f_{\mathrm{dm}};s+\ell-1)\prod_{p|D}(1-p^{-2s})\left(1-\overline{c(p)}(\Pi_p^{\sigma})^{\ell}p^{-s-\ell+1}\right)^{-1}.$$

□

8 An example

In this section, we present an example. We fix a positive integer ℓ divisible by w_K. Let $k=\ell$ and $\Xi=\mathbf{1}$. Take and fix a $\chi_2\in\mathcal{X}$ satisfying $\chi_0^{-1}\chi_2|_{\mathcal{O}_{K,f}^{\times}}=\mathbf{1}$ and $w_\infty(\chi_2)=2\ell-3$. Put $\eta=\chi_0^{-1}\chi_2$. Let $\varphi=\varphi_f\otimes\varphi_\infty\in\mathcal{S}(K_{\mathbf{A}})$ with $\varphi_f=\mathrm{char}_{\mathcal{O}_{K,f}}$ and $\varphi_\infty(z)=z^{\ell-2}\exp(-2\pi|z|^2)$ ($z\in K_\infty=\mathbf{C}$). Set

$$\theta_{\chi_2}(h)=\sum_{X\in K}\mathcal{M}_{\chi_2}^{T}(h)\varphi(X).$$

Lemma 8.1 *For $\ell>2$, we have $\theta_{\chi_2}\in S_{\ell-1}(D,\chi_0)$.*

Proof. It is easily seen that

$$\mathcal{M}_{\chi_2}^{T}(u_v)\varphi_v=\begin{cases}\widetilde{\chi_0}(u_f)\varphi_f & (v=f,\ u_f\in\mathcal{U}_0(D)_f),\\ J(u_\infty,i)^{1-\ell}\varphi_\infty & (v=\infty,\ u_\infty\in\mathcal{U}_\infty).\end{cases}$$

By these facts and Poisson summation formula, we see that

$$\theta_{\chi_2}(\gamma h u_f u_\infty) = \widetilde{\chi_0}(u_f) J(u_\infty, i)^{1-\ell} \theta_{\chi_2}(h) \quad (\gamma \in H_{\mathbf{Q}},\ h \in H_{\mathbf{A}},\ u_f \in \mathcal{U}_0(D)_f,\ u_\infty \in \mathcal{U}_\infty).$$

We next show that θ_{χ_2} is holomorphic. For $h_\infty = \boldsymbol{n}(x_\infty)\boldsymbol{d}(y_\infty) \in H_\infty$, we put $z = h_\infty \langle i \rangle = x_\infty + i\,\mathrm{N}(y_\infty) \in \mathfrak{H}$. Let $h_f \in H_{\mathbf{A},f}$. Then

$$\begin{aligned}
&J(h_\infty, i)^{\ell-1}\theta_{\chi_2}(h_\infty h_f) \\
&= (y_\infty^\sigma)^{1-\ell} \sum_{X \in K} \mathcal{M}^T_{\chi_2}(\boldsymbol{n}(x_\infty)\boldsymbol{d}(y_\infty))\varphi_\infty(X_\infty)\mathcal{M}^T_{\chi_2}(h_f)\varphi_f(X_f) \\
&= (y_\infty^\sigma)^{1-\ell}\chi_2(y_\infty)^{-1} \|y_\infty\|_{\mathbf{A}}^{1/2} \sum_{X \in K} \psi(x_\infty \mathrm{N}(X_\infty))\varphi_\infty(y_\infty X_\infty)\mathcal{M}^T_{\chi_2}(h_f)\varphi_f(X_f) \\
&= y_\infty^{2-\ell} \sum_{X \in K} \exp(2\pi i x_\infty \mathrm{N}(X_\infty))(y_\infty X_\infty)^{\ell-2} \exp(-2\pi |y_\infty X_\infty|^2)\mathcal{M}^T_{\chi_2}(h_f)\varphi_f(X_f) \\
&= \sum_{X \in K} X_\infty^{\ell-2} \exp(2\pi i\,\mathrm{N}(X_\infty) z)\mathcal{M}^T_{\chi_2}(h_f)\varphi_f(X_f),
\end{aligned}$$

which shows the claim. Finally we show that

$$\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} \theta_{\chi_2}(\boldsymbol{n}(x)h)dx = 0$$

for each $h \in H_{\mathbf{A}}$. We can put $h = \boldsymbol{d}(y_\infty)u_\infty h_f$ ($y_\infty \in \mathbf{C}^\times$, $u_\infty \in \mathcal{U}_\infty$, $h_f \in H_{\mathbf{A},f}$). Then we have

$$\begin{aligned}
\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} \theta_{\chi_2}(\boldsymbol{n}(x)h)dx &= \int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} \sum_{X \in K} \psi(x\,\mathrm{N}(X))\mathcal{M}^T_{\chi_2}(h)\varphi(X)dx \\
&= \mathcal{M}^T_{\chi_2}(h)\varphi(0) \\
&= J(u_\infty, i)^{1-\ell}\chi_2(y_\infty)^{-1} \|y_\infty\|_{\mathbf{A}}^{1/2} \varphi_\infty(0)\mathcal{M}^T_{\chi_2}(h_f)\varphi_f(0) \\
&= 0.
\end{aligned}$$

Hence we obtain $\theta_{\chi_2} \in S_{\ell-1}(D, \chi_0)$. □

For $\Omega \in \mathcal{Y}_\ell$ with $\ell > 2$, let $\widetilde{\Omega}(z) = \Omega(z/z^\sigma)$ ($z \in K_{\mathbf{A}}^\times$), and put

$$\Theta_\Omega(h) = \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1})\theta_{\chi_2}(th)d^1t.$$

Here $d^1t = \prod_{v \le \infty} d^1t_v$ is the Haar measure on $K^1_{\mathbf{A}}$ normalized by $\int_{\mathcal{O}^1_{K,p}} d^1t_p = \int_{K^1_\infty} d^1t_\infty = 1$. Then Θ_Ω is in $S_{\ell-1}(D, \chi_0; \chi_0\Omega)$.

Lemma 8.2 Θ_Ω *is a Hecke eigenform with eigenvalues* $\{\Lambda_p\}$, *where*

$$\begin{aligned}\Lambda_p &= \begin{cases} p+1 & (p \ \textit{is inert in}\, K/\mathbf{Q}), \\ p^{1/2}\left\{\eta(\Pi_p)+\eta(\Pi_p)^{-1}\right\} & (p \ \textit{ramifies in}\, K/\mathbf{Q}), \end{cases} \\ \Lambda_{p,j} &= p^{1/2}\left\{\eta(\Pi_{p,j})+(\eta^{-1}\widetilde{\Omega})(\Pi_{p,j})\right\} \quad (p \ \textit{splits in}\, K/\mathbf{Q},\ j=1,2).\end{aligned}$$

Proof. We first suppose that p is inert. Then we have

$$\begin{aligned} &-\mathcal{M}_{\chi_2}^T(\boldsymbol{d}(p^{-1}))\varphi_p(X) - \sum_{x\in \mathbf{Z}_p^\times/p\mathbf{Z}_p}\mathcal{M}_{\chi_2}^T(\boldsymbol{n}(p^{-1}x))\varphi_p(X) - \sum_{y\in \mathbf{Z}_p/p^2\mathbf{Z}_p}\mathcal{M}_{\chi_2}^T(\boldsymbol{n}(y)\boldsymbol{d}(p))\varphi_p(X) \\ &= -\chi_{2,p}(p^{-1})^{-1}\left\|p^{-1}\right\|_p^{1/2}\varphi_p(p^{-1}X) - \sum_{x\in \mathbf{Z}_p^\times/p\mathbf{Z}_p}\psi_p(p^{-1}x\,\mathrm{N}(X))\varphi_p(X) \\ &\quad - \sum_{y\in \mathbf{Z}_p/p^2\mathbf{Z}_p}\chi_{2,p}(p)^{-1}\left\|p\right\|_p^{1/2}\psi_p(y\,\mathrm{N}(X))\varphi_p(pX) \\ &= -\chi_{2,p}(p)p\varphi_p(p^{-1}X) - \left\{p\varphi_p(p^{-1}X)-1\right\}\varphi_p(X) - \chi_{2,p}(p)^{-1}p^{-1}p^2\varphi_p(X) \\ &= p\varphi_p(p^{-1}X) - p\varphi_p(p^{-1}X) + \varphi_p(X) + p\varphi_p(X) \\ &= (p+1)\varphi_p(X). \end{aligned}$$

Hence we see that

$$\mathcal{T}_p\Theta_\Omega(h) = (p+1)\Theta_\Omega(h).$$

Next suppose that p ramifies. Put

$$\begin{aligned} I^+(X) &= \chi_{0,p}(\Pi_p)\sum_{y\in \mathbf{Z}_p/p\mathbf{Z}_p}\mathcal{M}_{\chi_2}^T(\boldsymbol{n}(y)\boldsymbol{d}(\Pi_p))\varphi_p(X), \\ I^-(X) &= \chi_{0,p}(\Pi_p)^{-1}\sum_{x\in \mathbf{Z}_p/p\mathbf{Z}_p}\mathcal{M}_{\chi_2}^T(\overline{\boldsymbol{n}}(Dx)\boldsymbol{d}(\Pi_p^{-1}))\varphi_p(X). \end{aligned}$$

We show that

$$I^{\pm}(X) = p^{1/2}\eta(\Pi_p)^{\mp 1}\varphi_p(X),$$

which proves the claim. First we have

$$\begin{aligned} I^+(X) &= \chi_{0,p}(\Pi_p)\chi_{2,p}(\Pi_p)^{-1}\left\|\Pi_p\right\|_p^{1/2}\sum_{y\in \mathbf{Z}_p/p\mathbf{Z}_p}\psi_p(y\,\mathrm{N}(X))\varphi_p(\Pi_pX) \\ &= p^{-1/2}\eta(\Pi_p)^{-1}p\varphi_p(X)\varphi_p(\Pi_pX) \\ &= p^{1/2}\eta(\Pi_p)^{-1}\varphi_p(X). \end{aligned}$$

We next obtain

$$
\begin{aligned}
& I^{-}(X) \\
= \; & \omega_p(-1)\chi_{0,p}(\Pi_p)^{-1} \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \int_{K_p} \psi_{K_p}(Y^\sigma X)\psi_p(-Dx\,\mathrm{N}(Y)) \\
& \times \left\{ \int_{K_p} \psi_{K_p}(Z^\sigma Y)\chi_{2,p}(-\Pi_p^{-1})^{-1} \left\| -\Pi_p^{-1} \right\|_p^{1/2} \varphi_p(-\Pi_p^{-1} Z) dZ \right\} dY \\
= \; & p^{1/2}\eta(\Pi_p) \\
& \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \int_{K_p} \psi_{K_p}(Y^\sigma X)\psi_p(-Dx\,\mathrm{N}(Y)) \left\{ \|\Pi_p\|_p \int_{\mathcal{O}_{K,p}} \psi_{K_p}(\Pi_p^\sigma Z^\sigma Y) dZ \right\} dY \\
= \; & p^{1/2}\eta(\Pi_p) \\
& \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} p^{-1} |D|_p^{1/2} \int_{K_p} \psi_{K_p}(Y^\sigma X)\psi_p(-Dx\,\mathrm{N}(Y))\varphi_p(\sqrt{D}\Pi_p^\sigma Y) dY \\
= \; & p^{1/2}\eta(\Pi_p) \\
& \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} p^{-1} |D|_p^{1/2} \left\| \sqrt{D}\Pi_p^\sigma \right\|_p^{-1} \int_{\mathcal{O}_{K,p}} \psi_{K_p}(-\sqrt{D}^{-1}\Pi_p^{-1} X Y^\sigma)\psi_p(\mathrm{N}(\Pi_p)^{-1} x\,\mathrm{N}(Y)) dY \\
= \; & p^{1/2}\eta(\Pi_p) p |D|_p^{-1/2} \int_{\Pi_p \mathcal{O}_{K,p}} \psi_{K_p}(-\sqrt{D}^{-1}\Pi_p^{-1} X Y^\sigma) dY \\
= \; & p^{1/2}\eta(\Pi_p) p |D|_p^{-1/2} \|\Pi_p\|_p \int_{\mathcal{O}_{K,p}} \psi_{K_p}(-\sqrt{D}^{-1}\Pi_p^{-1}\Pi_p^\sigma X Y^\sigma) dY \\
= \; & p^{1/2}\eta(\Pi_p)\varphi_p(X).
\end{aligned}
$$

Finally suppose that p splits. For $j = 1, 2$, put

$$
I_j(X) = \chi_{0,p}(\Pi_{p,j})^{-1} \left\{ \mathcal{M}_{\chi_2}^T(\boldsymbol{d}(\Pi_{p,j}^{-1}))\varphi_p(X) + \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \mathcal{M}_{\chi_2}^T(\boldsymbol{n}(x)\boldsymbol{d}(\Pi_{p,j}^\sigma))\varphi_p(X) \right\}.
$$

Then we have

$$
\begin{aligned}
I_j(X) &= p^{1/2}\eta(\Pi_{p,j})\varphi_p(\Pi_{p,j}^{-1}X) + p^{-1/2}\eta(\Pi_{p,j})\varphi_p(\Pi_{p,j}^\sigma X) \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \psi_p(x\,\mathrm{N}(X)) \\
&= p^{1/2}\eta(\Pi_{p,j}) \left\{ \varphi_p(\Pi_{p,j}^{-1}X) + \mathrm{char}_{\mathbf{Z}_p}(\mathrm{N}(X))\varphi_p(\Pi_{p,j}^\sigma X) \right\}.
\end{aligned}
$$

Now it is easily seen that

$$\begin{aligned}
&\mathrm{char}_{\mathbf{Z}_p}(\mathrm{N}(X))\varphi_p(\Pi_{p,1}^{\sigma}X) \\
=\ & \sum_{m+n\geq 0} \mathrm{char}_{\Pi_{p,1}^m\Pi_{p,2}^n\mathcal{O}_{K,p}^{\times}}(X)\,\mathrm{char}_{\Pi_{p,2}^{-1}\mathcal{O}_{K,p}}(X) \\
=\ & \sum_{\substack{m+n\geq 0\\ m\geq 0,\, n\geq -1}} \mathrm{char}_{\Pi_{p,1}^m\Pi_{p,2}^n\mathcal{O}_{K,p}^{\times}}(X) \\
=\ & \sum_{m,n\geq 0} \mathrm{char}_{\Pi_{p,1}^m\Pi_{p,2}^n\mathcal{O}_{K,p}^{\times}}(X) + \sum_{m\geq 1,\, n=-1} \mathrm{char}_{\Pi_{p,1}^m\Pi_{p,2}^n\mathcal{O}_{K,p}^{\times}}(X) \\
=\ & \varphi_p(X) + \sum_{m\geq 1,\, n\geq -1} \mathrm{char}_{\Pi_{p,1}^m\Pi_{p,2}^n\mathcal{O}_{K,p}^{\times}}(X) - \sum_{m\geq 1,\, n\geq 0} \mathrm{char}_{\Pi_{p,1}^m\Pi_{p,2}^n\mathcal{O}_{K,p}^{\times}}(X) \\
=\ & \varphi_p(X) + \varphi_p(\Pi_{p,1}^{-1}\Pi_{p,2}X) - \varphi_p(\Pi_{p,1}^{-1}X).
\end{aligned}$$

Similarly we obtain

$$\mathrm{char}_{\mathbf{Z}_p}(\mathrm{N}(X))\varphi_p(\Pi_{p,2}^{\sigma}X) = \varphi_p(X) + \varphi_p(\Pi_{p,2}^{-1}\Pi_{p,1}X) - \varphi_p(\Pi_{p,2}^{-1}X).$$

Hence we get

$$I_j(X) = p^{1/2}\eta(\Pi_{p,j})\left\{\varphi_p(X) + \varphi_p(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma}X)\right\}.$$

Therefore we have

$$\begin{aligned}
&\mathcal{T}_{p,j}\Theta_{\Omega}(h) \\
=\ & \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1})\mathcal{T}_{p,j}\theta_{\chi_2}(th)d^1t \\
=\ & p^{1/2}\eta(\Pi_{p,j}) \\
& \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1})\left\{\sum_{X\in K}\mathcal{M}_{\chi_2}^T(th)\varphi_p(X) + \sum_{X\in K}\mathcal{M}_{\chi_2}^T(th)\varphi_p(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma}X)\right\}d^1t \\
=\ & p^{1/2}\eta(\Pi_{p,j})\left\{\Theta_{\Omega}(h) + \chi_{2,p}(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma})\Theta_{\Omega}(\boldsymbol{d}(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma})h)\right\} \\
=\ & p^{1/2}\eta(\Pi_{p,j})\left\{\Theta_{\Omega}(h) + \eta(\Pi_{p,j}^{-1}\Pi_{p,j}^{\sigma})\widetilde{\Omega}(\Pi_{p,j})\Theta_{\Omega}(h)\right\} \\
=\ & p^{1/2}\left\{\eta(\Pi_{p,j}) + (\eta^{-1}\widetilde{\Omega})(\Pi_{p,j})\right\}\Theta_{\Omega}(h).
\end{aligned}$$

This completes the proof. □

Lemma 8.3 *We have*

$$\mathfrak{F}_{D,p}\Theta_{\Omega} = \varepsilon_p\Theta_{\Omega} \quad \textit{with} \quad \varepsilon_p = \lambda_{K,p}(\psi_p)^{-1}\chi_{2,p}(\sqrt{D})$$

for each $p \mid D$.

Proof. Let $p \mid D$. Then

$$\begin{aligned}
&\mathcal{M}^T_{\chi_2}(w_{D,p})\varphi_p(X) \\
&= \mathcal{M}^T_{\chi_2}(\boldsymbol{d}(-\sqrt{D}^{-1})_p S_p)\varphi_p(X) \\
&= \lambda_{K,p}(\psi_p)\chi_{2,p}(-\sqrt{D}^{-1})^{-1}\left\|-\sqrt{D}^{-1}\right\|_p^{1/2}\int_{K_p}\psi_{K_p}(-\sqrt{D}^{-1}XY_p^\sigma)\varphi_p(Y_p)dY_p \\
&= \lambda_{K,p}(\psi_p)\chi_{2,p}(-\sqrt{D})\,|D|_p^{-1/2}\int_{\mathcal{O}_{K,p}}\psi_{K_p}(-\sqrt{D}^{-1}XY_p^\sigma)dY_p \\
&= \lambda_{K,p}(\psi_p)\chi_{2,p}(-\sqrt{D})\varphi_p(X) \\
&= \lambda_{K,p}(\psi_p)^{-1}\chi_{2,p}(\sqrt{D})\varphi_p(X).
\end{aligned}$$

This shows

$$\mathfrak{F}_{D,p}\Theta_\Omega(h) = \Theta_\Omega(hw_{D,p}) = \lambda_{K,p}(\psi_p)^{-1}\chi_{2,p}(\sqrt{D})\Theta_\Omega(h),$$

and we are done. □

Lemma 8.4 *We have*

$$L(\Theta_\Omega, \mathbf{1}; s) = L(\eta; s)L(\eta^{-1}\widetilde{\Omega}; s).$$

Here $L(\eta; s)$ is the Hecke L-function attached to the Hecke character η.

Proof. Suppose that p is inert. Then we have

$$\begin{aligned}
L_p(\Theta_\Omega, \mathbf{1}; s) &= \left(1 - 2p^{-2s} + p^{-4s}\right)^{-1} \\
&= \left(1 - \eta(p)p^{-2s}\right)^{-1}\left(1 - (\eta^{-1}\widetilde{\Omega})(p)p^{-2s}\right)^{-1} \\
&= \left(1 - \eta(p)\,|\mathrm{N}(p)|_p^{-s}\right)^{-1}\left(1 - (\eta^{-1}\widetilde{\Omega})(p)\,|\mathrm{N}(p)|_p^{-s}\right)^{-1}.
\end{aligned}$$

Note that $\eta(p) = \widetilde{\Omega}(p) = 1$. Next suppose that p splits. Then it is easily seen that

$$\begin{aligned}
L_p(\Theta_\Omega, \mathbf{1}; s) &= \prod_{j=1,2}\left(1 - \left\{\eta(\Pi_{p,j}) + (\eta^{-1}\widetilde{\Omega})(\Pi_{p,j})\right\}p^{-s} + \widetilde{\Omega}(\Pi_{p,j})p^{-2s}\right)^{-1} \\
&= \prod_{j=1,2}\left(1 - \eta(\Pi_{p,j})\,|\mathrm{N}(\Pi_{p,j})|_p^{-s}\right)^{-1}\left(1 - (\eta^{-1}\widetilde{\Omega})(\Pi_{p,j})\,|\mathrm{N}(\Pi_{p,j})|_p^{-s}\right)^{-1}.
\end{aligned}$$

Finally suppose that p ramifies. Then we obtain

$$\begin{aligned}
L_p(\Theta_\Omega, \mathbf{1}; s) &= \left(1 - \left\{\eta(\Pi_p) + \eta(\Pi_p)^{-1}\right\}p^{-s} + p^{-2s}\right)^{-1} \\
&= \left(1 - \eta(\Pi_p)\,|\mathrm{N}(\Pi_p)|_p^{-s}\right)^{-1}\left(1 - (\eta^{-1}\widetilde{\Omega})(\Pi_p)\,|\mathrm{N}(\Pi_p)|_p^{-s}\right)^{-1}.
\end{aligned}$$

Note that $\widetilde{\Omega}(\Pi_p) = 1$. Hence we see that

$$L(\Theta_\Omega, \mathbf{1}; s) = \prod_{p<\infty} L_p(\Theta_\Omega, \mathbf{1}; s) = L(\eta; s) L(\eta^{-1}\widetilde{\Omega}; s).$$

□

Proposition 8.5 *We have*

$$L^*(\Theta_\Omega, \mathbf{1}; s) = L^*(\Theta_\Omega, \mathbf{1}; 1-s).$$

Proof. We first calculate the values of $W_{\Theta_\Omega}(I)$ and $W_{\Theta_\Omega}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A))$ for $p = 2$ and $A = 2, 4$. Since $\theta_{\chi_2}(t\boldsymbol{n}(x)) = \chi_2(t) \sum_{X \in K} \psi(x \,\mathrm{N}(X)) \varphi(t^{-1}X)$, we have

$$\begin{aligned}
W_{\Theta_\Omega}(I) &= \int_{\mathbf{Q}\backslash\mathbf{Q_A}} \psi(-x) \left\{ \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1}) \theta_{\chi_2}(t\boldsymbol{n}(x)) d^1 t \right\} dx \\
&= \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1}) \chi_2(t) \sum_{X \in K^1} \varphi(t^{-1}X) d^1 t \\
&= \int_{K^1_{\mathbf{A}}} \eta(t) \Omega(t^{-1}) \varphi(t^{-1}) d^1 t \\
&= \prod_{v \le \infty} \int_{K^1_v} \eta(t_v) \Omega(t_v^{-1}) \varphi_v(t_v^{-1}) d^1 t_v \\
&= e^{-2\pi} \int_{K^1_\infty} d^1 t_\infty \int_{\mathcal{O}^1_{K,f}} d^1 t_f \\
&= e^{-2\pi}.
\end{aligned}$$

Let $p = 2$ and $A = p^{\mathrm{ord}_p D - 1} \in \mathbf{Z}_p$. We obtain

$$\begin{aligned}
&W_{\Theta_\Omega}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A)) \\
&= \int_{\mathbf{Q}\backslash\mathbf{Q_A}} \psi(-x) \left\{ \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1}) \theta_{\chi_2}(t\boldsymbol{n}(x)\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A)) d^1 t \right\} dx \\
&= \int_{\mathbf{Q}\backslash\mathbf{Q_A}} \psi(-x) \left\{ \int_{K^1\backslash K^1_{\mathbf{A}}} (\chi_0\Omega)(t^{-1}) \chi_2(t) \psi(x \,\mathrm{N}(X)) \sum_{X \in K} \mathcal{M}^T_{\chi_2}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A)) \varphi(t^{-1}X) d^1 t \right\} dx \\
&= \int_{K^1\backslash K^1_{\mathbf{A}}} \eta(t) \Omega(t^{-1}) \sum_{X \in K^1} \mathcal{M}^T_{\chi_2}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A)) \varphi(t^{-1}X) d^1 t \\
&= \int_{K^1_{\mathbf{A}}} \eta(t) \Omega(t^{-1}) \mathcal{M}^T_{\chi_2}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A)) \varphi(t^{-1}) d^1 t.
\end{aligned}$$

Since

$$\mathcal{M}^T_{\chi_2}(\overline{\boldsymbol{n}}(A))\varphi_p(Y) = |D|_p^{-1/2}\, I_p(A,Y) = \mathrm{char}_{\Pi_p^{-1}\mathcal{O}^\times_{K,p}}(Y)$$

by Lemma 6.6 and 6.7, we have

$$\begin{aligned}\mathcal{M}^T_{\chi_2}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A))\varphi_p(t^{-1}) &= \sqrt{p}\chi_{2,p}(\Pi_p)\mathcal{M}^T_{\chi_2}(\overline{\boldsymbol{n}}(A))\varphi_p(\Pi_p^{-1}t^{-1}) \\ &= \sqrt{p}\chi_{2,p}(\Pi_p)\,\mathrm{char}_{\mathcal{O}^\times_{K,p}}(t^{-1}).\end{aligned}$$

Hence we get

$$\begin{aligned}W_{\Theta_\Omega}(\boldsymbol{d}(\Pi_p^{-1})\overline{\boldsymbol{n}}(A)) &= e^{-2\pi}\sqrt{p}\chi_{2,p}(\Pi_p)\int_{K^1_\infty} d^1t_\infty \int_{\mathcal{O}^1_{K,f}} d^1t_f \\ &= e^{-2\pi}\sqrt{p}\chi_{2,p}(\Pi_p).\end{aligned}$$

Next we show that $\mathfrak{C}_2(\Theta_\Omega) \neq 0$ (for the definition, see (4.2)). We first suppose that $\mathrm{ord}_2 D = 2$. Note that $\varepsilon_2 = \lambda_{K,2}(\psi_2)^{-1}\chi_{2,2}(\sqrt{D}) = i\chi_{0,2}(\sqrt{D})$. Then we have

$$\begin{aligned}\mathfrak{C}_2(\Theta_\Omega) &= \chi_{0,2}(\Pi_2)^{-1}W_{\Theta_\Omega}(\boldsymbol{d}(\Pi_2^{-1})\overline{\boldsymbol{n}}(2))\Lambda_2 \\ &= 2e^{-2\pi}\eta(\Pi_2)\left\{\eta(\Pi_2)+\eta(\Pi_2)^{-1}\right\} \\ &= 4e^{-2\pi} \quad (\neq 0).\end{aligned}$$

Here we used the fact $\eta(\Pi_2)^2 = 1$. Next we suppose that $\mathrm{ord}_2 D = 3$. Since $\varepsilon_2 = \lambda_{K,2}(\psi_2)\chi_{2,2}(\sqrt{D})^{-1}$, we see that

$$\Lambda_2 - \sqrt{2}\varepsilon_2\chi_{0,2}(\sqrt{D})\lambda_{K,2}(\psi_2)^{-1} = \sqrt{2}\left\{\eta(\Pi_2)+\eta(\Pi_2)^{-1}\right\} - \sqrt{2}\eta(\sqrt{D})^{-1} = \sqrt{2}\eta(\Pi_2).$$

Note that $\eta(\sqrt{D}) = \eta(\Pi_2)$. Hence we get

$$\begin{aligned}\mathfrak{C}_2(\Theta_\Omega) &= \chi_{0,2}(\Pi_2)^{-1}W_{\Theta_\Omega}(\boldsymbol{d}(\Pi_2^{-1})\overline{\boldsymbol{n}}(4))\sqrt{2}\eta(\Pi_2) \\ &= 2e^{-2\pi}\eta(\Pi_2)^2 \\ &= 2e^{-2\pi} \quad (\neq 0).\end{aligned}$$

In the remaining case, it is easily seen that

$$\mathfrak{C}_2(\Theta_\Omega) = W_{\Theta_\Omega}(I) = e^{-2\pi} \quad (\neq 0).$$

Hence we have $\mathfrak{C}_2(\Theta_\Omega) \neq 0$. Therefore Θ_Ω satisfies the assumptions of Corollary 4.3, and we have

$$L^*(\Theta_\Omega, \mathbf{1}; s) = L^*(\Theta_\Omega, \mathbf{1}; 1-s).$$

□

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次数 2 のユニタリ群上の尖点形式に付随する L-関数の解析的性質について

門　田　智　則

概　要

fをユニタリ群 $U(1,1)$ 上の正則な尖点形式とする．この論文では，fに付随する標準的な L-関数を研究し，その関数等式をある条件の下で示す．