When do the harmonic Hardy spaces with distinct indices coincide on a hyperbolic Riemann surface?

Dedicated to Professor Hisashi Ishida on his sixtieth birthday

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Abstract

Let R be a hyperbolic Riemann surface. Suppose that $1 \leq p < q \leq \infty$. In this paper we give a characterization that two harmonic Hardy spaces $h_p(R)$ and $h_q(R)$ coincide with each other by using the term of the Martin boundary Δ^M of R. Let Δ_1^M be the minimal Martin boundary of R. In the case that $p > 1$ it holds that $h_p(R)$ coincides with $h_q(R)$ if and only if there exists a nullset N of Δ^M with respect to the harmonic measure such that $\Delta_1^M \setminus N$ consists of finitely many points with positive harmonic measure. In the case that $p = 1$ it holds that $h_1(R)$ coincides with $h_q(R)$ if and only if Δ_1^M consists of finitely many points with positive harmonic measure.

Keywords: hyperbolic Riemann surface, harmonic Hardy space, Martin boundary, minimal Martin boundary, harmonic measure

1. Introduction

Denote by O_G the class of open Riemann surfaces R such that there exist no Green's functions on R. We say that an open Riemann surface R is *parabolic* (resp. *hyperbolic*) if R belongs (resp. does not belong) to O_G.

For an open Riemann surface R, we denote by $HP_+(R)$ and $HB_+(R)$ the classes of *non-negative* harmonic functions and *non-negative bounded* harmonic functions on R, respectively. Denote by $MHB_+(R)$ the class of all finite limit functions of monotone increasing sequences of $HB_+(R)$. Set $HX(R) = HX_+(R) - HX_+(R)$ $(X = P, B)$, where $HX_+(R) - HX_+(R) = \{h_1 - h_2 \mid h_j \in HX_+(R)\ (j = 1, 2)\ \}$, and $MHB(R) = MHB_+(R) - MHB_+(R)$. Then, $HB(R)$ are the class of *bounded* harmonic functions on R. $MHB(R)$ is called the class of *quasi-bounded* functions on R. It is well-known that if R is parabolic, then $HX(R)$ $(X = P, B)$ and $MHB(R)$ consist of constant functions (cf. [4]).

Hereafter, we consider only hyperbolic Riemann surfaces R. Let $\Delta^M = \Delta^{R,M}$ and $\Delta_1^M = \Delta_1^{R,M}$ the *Martin boundary* of R and the *minimal Martin boundary* on R, respectively. We refer to [1] for the details about the Martin boundary. Denote by $h_p(R)$ $(1 \leq p \leq \infty)$ the harmonic Hardy space with index p on R (see Definition of harmonic Hardy space in the next section). It is well-known that, if $1 \leq p < q, h_q(R) \subset h_p(R)$. It is natural to ask when the converse inclusion relation holds. The purpose of this paper is to answer the question.

Theorem 1. *Suppose that R is hyperbolic and $1 < p < q \leq \infty$. Then the followings are equivalent*:

(i) $h_p(R) = h_q(R)$,

(ii) *there exists a nullset N of Δ^M with respect to the harmonic measure such that $\Delta_1^M \backslash N$ consists of finitely many points with positive harmonic measure,*

(iii) $\dim h_p(R) = \dim h_q(R) < \infty$,

where $\dim h_p(R)$ *is the dimension of the linear space* $h_p(R)$.

Theorem 2. *Suppose that R is hyperbolic, $p = 1$ and $1 < q \leq \infty$. Then the followings are equivalent*:

(i) $h_1(R) = h_q(R)$,

(ii) Δ_1^M *consists of finitely many points with positive harmonic measure,*

(iii) $\dim h_1(R) = \dim h_q(R) < \infty$,

where $\dim h_1(R)$ *is the dimension of the linear space* $h_1(R)$.

As an immediate consequence of Theorem 2, by the fact that $h_1(R) = HP(R)$ and definition of $h_\infty(R)$, we obtain the following.

Corollary (cf. [2, Theorem]). *Suppose that R is hyperbolic. Then the followings are equivalent*:

(i) $HP(R) = HB(R)$,

(ii) Δ_1^M *consists of finitely many points with positive harmonic measure,*

(iii) $\dim HP(R) = \dim HB(R) < \infty$,

where $\dim HP(R)$ *is the dimension of the linear space* $HP(R)$.

2. Preliminaries

In this section we state several propositions in order to prove theorems in §1 in the next section. Denote by $\omega_z^M(\cdot)$ the *harmonic measure* on Δ^M with respect to $z \in R$. We also denote by $k_\zeta(z)$ $((\zeta, z) \in (R \cup \Delta^M) \times R)$ the *Martin kernel* on R with pole at ζ. The following proposition plays fundamental role in the proof of Theorem 2.

Proposition 1 (cf. [1, Hilfssatz 13.3]). *Let ζ belong to Δ_1^M. Then the Martin kernel $k_\zeta(\cdot)$ with pole at ζ is bounded on R if and only if the harmonic measure $\omega_\cdot(\{\zeta\})$ of the singleton $\{\zeta\}$ is positive.*

The next proposition follows from the Martin representation theorem which is the most fundamental theorem in the Martin theory.

Proposition 2. *Let u be an element in $HP(R)$. There exists a signed measure μ on Δ^M such that $\mu(\Delta^M \setminus \Delta_1^M) = 0$ and $u = \int_{\Delta_1^M} k_\xi d\mu(\xi)$.*

Proof of Proposition 2. Let u be an element in $HP(R)$. By definition of $HP(R)$ there exist elements u_1 and u_2 of $HP_+(R)$ with $u = u_1 - u_2$ on R. By the Martin representation theorem(cf. [1, Satz 13.1]) we find the Radon measure μ_j $(j = 1, 2)$ on Δ^M such that $\mu_j(\Delta^M \setminus \Delta_1^M) = 0$ $(j = 1, 2)$

and $u_j = \int_{\Delta_1^M} k_\xi d\mu_j(\xi)$ $(j = 1, 2)$. Set $\mu = \mu_1 - \mu_2$. Then μ is a signed measure on Δ^M. We have

$$\mu(\Delta^M \setminus \Delta_1^M) = \mu_1(\Delta^M \setminus \Delta_1^M) - \mu_2(\Delta^M \setminus \Delta_1^M) = 0 - 0 = 0$$

and

$$u = u_1 - u_2 = \int_{\Delta_1^M} k_\xi d\mu_1(\xi) - \int_{\Delta_1^M} k_\xi d\mu_2(\xi) = \int_{\Delta_1^M} k_\xi d\mu(\xi).$$

We have the desired result.

Definition (cf. [3, Definition in p.437 and Theorem 4])**.** Let $p \geq 1$. Set

$$h_p(R) = \begin{cases} \{u \mid u \text{ is harmonic on } R \text{ and } |u|^p \text{ has a harmonic majorant on } R\}, & \text{for } p \geq 1, \\ HB(R), & \text{for } p = \infty. \end{cases}$$

We call $h_p(R)$ the *harmonic Hardy space* with index p on R. We remark that $h_1(R) = HP(R)$ and that, if $1 \leq p < q \leq \infty$, $h_q(R) \subset h_p(R)$.

Proposition 3 (cf. [3, Definition in p.437 and Theorems 4 and 6])**.** *Let p be a real number with $1 < p < \infty$. Fix a point z_0 of R. The next conditions are equivalent.*

(i) $u \in h_p(R)$,

(ii) *u has the minimal fine limit $u^*(\zeta)$ at almost every point $\zeta (\in \Delta_1^M)$ with respect to the harmonic measure $\omega_{z_0}^M$ such that $u(z) = \int_{\Delta_1^M} u^*(\zeta) d\omega_z^M(\zeta)$, and $\int_{\Delta_1^M} |u^*(\zeta)|^p d\omega_{z_0}^M(\zeta) < \infty$.*

Set $h_{p+}(R) := h_p(R) \cap HP_+(R)$. By the above proposition it is easily seen that $h_p(R) = h_{p+}(R) - h_{p+}(R)$ and $h_{p+}(R) \subset MHB_+(R)$.

3. Proof of Theorems

3.1 Proof of Theorem 1

First we consider $q \neq \infty$. Let p and q be real numbers with $1 < p < q$. Suppose that (i) holds. Fix a point z_0 of R. Further we suppose that there exists a point $\zeta \in \Delta^M$ such that, for any positive ρ, $\omega_{z_0}^M(U_\rho(\zeta)) > 0$ and $\omega_{z_0}^M(\{\zeta\}) = 0$, where $U_\rho(\zeta)$ is the disc with center ζ and radius ρ with respect to the standard metric on $R \cup \Delta_1^M$. Hence, there exists a monotone decreasing sequence $\{\rho_n\}$ with $\lim_{n\to\infty} \rho_n = 0$, $\omega_{z_0}^M(U_{\rho_n}(\zeta) \setminus U_{\rho_{n+1}}(\zeta)) > 0$ and $\omega_{z_0}^M(U_{\rho_n}(\zeta)) \leq 1/n^{2q/(q-p)}$ $(n \in \mathbb{N})$. Set

$$f^*(\xi) = \begin{cases} [\omega_{z_0}^M(U_{\rho_n}(\zeta) \setminus U_{\rho_{n+1}}(\zeta))]^{-1/q}, & \text{for } \xi \in U_{\rho_n}(\zeta) \setminus U_{\rho_{n+1}}(\zeta), \\ 0, & \text{for } \xi \in \Delta^M \setminus U_{\rho_1}(\zeta). \end{cases}$$

And set $f(z) = \int_{\Delta_1^M} f^*(\xi) d\omega_z^M(\xi)$. Then, we find that f has a minimal fine limit $f^*(\xi)$ at almost every point $\xi \in \Delta_1^M$ with respect to $\omega_{z_0}^M$. Hence, we have

$$\begin{aligned} \int_{\Delta_1^M} f^*(\xi)^q d\omega_{z_0}^M(\xi) &= \sum_{n=1}^{\infty} [\omega_{z_0}^M(U_{\rho_n}(\zeta) \setminus U_{\rho_{n+1}}(\zeta))]^{-1} \omega_{z_0}^M(U_{\rho_n}(\zeta) \setminus U_{\rho_{n+1}}(\zeta)) \\ &= \sum_{n=1}^{\infty} 1 = \infty, \end{aligned}$$

and

$$\begin{aligned}\int_{\Delta_1^M} f^*(\xi)^p d\omega_{z_0}^M(\xi) &= \sum_{n=1}^{\infty}[\omega_{z_0}^M(U_{\rho_n}(\zeta)\setminus U_{\rho_{n+1}}(\zeta))]^{-p/q}\omega_{z_0}^M(U_{\rho_n}(\zeta)\setminus U_{\rho_{n+1}}(\zeta))\\ &= \sum_{n=1}^{\infty}[\omega_{z_0}^M(U_{\rho_n}(\zeta)\setminus U_{\rho_{n+1}}(\zeta))]^{1-p/q}\\ &\le \sum_{n=1}^{\infty}[\omega_{z_0}^M(U_{\rho_n}(\zeta)\setminus U_{\rho_{n+1}}(\zeta))]^{(q-p)/q}\\ &\le \sum_{n=1}^{\infty} 1/n^2 < \infty.\end{aligned}$$

By Proposition 3 we find that $f \in h_p(R)\setminus h_q(R)$. By (i) this is a contradiction. Hence, if $\zeta(\in \Delta^M)$ satisfies that, for any positive $\rho, \omega_{z_0}^M(U_\rho(\zeta)) > 0, \omega_{z_0}^M(\{\zeta\}) > 0$. By the above fact, it holds that there exists a nullset N of Δ^M with respect to $\omega_{z_0}^M$ such that $\Delta_1^M \setminus N$ consists of at most countably many points with positive harmonic measure. To see this set

$$N = \{\zeta \in \Delta^M |\ \text{there exists a positive } \rho_\zeta \text{ with } \omega_{z_0}^M(U_{\rho_\zeta}(\zeta)) = 0\}$$

and set $F = \Delta^M \setminus N$. Clearly $F \cup N = \Delta^M, F \cap N = \emptyset$ and, for any $\zeta \in F, \omega_{z_0}^M(\{\zeta\}) > 0$. Hence F is an at most countable subset of Δ_1^M because $\omega_{z_0}^M(\Delta^M \setminus \Delta_1^M) = 0$. Hence it is sufficient to prove that $\omega_{z_0}^M(N) = 0$. Set $O = \cup_{\zeta\in N} U_{\rho_\zeta}(\zeta)$. Clearly O is an open subset of $R \cup \Delta^M$ and $O \cap \Delta^M = N$. By the Lindelöf theorem there exists a sequence $\{\xi_n\}_{n=1}^\infty$ of N with $O = \cup_{n=1}^\infty U_{\rho_{\xi_n}}(\xi_n)$. Hence $\omega_{z_0}^M(N) \le \omega_{z_0}^M(O) \le \sum_{n=1}^\infty \omega_{z_0}^M(U_{\rho_{\xi_n}}(\xi_n)) = 0$, and hence, $\omega_{z_0}^M(N) = 0$.

Suppose that $\sharp(\Delta_1^M\setminus N) = \aleph_0$, where $\sharp(\Delta_1^M\setminus N)$ is the cardinal number of $\Delta_1^M\setminus N$. Set $\Delta_1^M\setminus N = \{\zeta_n\}_{n=1}^\infty$. Hence there exists a subsequence $\{\zeta_{n_l}\}_{l=1}^\infty$ of $\{\zeta_n\}$ with $\omega_{z_0}^M(\{\zeta_{n_l}\}) \le 1/l^{2q/(q-p)}$ $(l \in \mathbb{N})$. Set

$$g^*(\xi) = \begin{cases}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{-1/q}, & \text{for } \xi = \zeta_{n_l},\\ 0, & \text{for } \xi \in \Delta^M \setminus \{\zeta_{n_l}\}_{l=1}^\infty.\end{cases}$$

And set $g(z) = \int_{\Delta_1^M} g^*(\xi)d\omega_z^M(\xi)$. Then, we find that g has a minimal fine limit $g^*(\xi)$ at almost every point $\xi \in \Delta_1^M$ with respect to $\omega_{z_0}^M$. Hence, we have

$$\begin{aligned}\int_{\Delta_1^M} g^*(\xi)^q d\omega_{z_0}^M(\xi) &= \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{-1}\omega_{z_0}^M(\{\zeta_{n_l}\})\\ &= \sum_{n=1}^{\infty} 1 = \infty,\end{aligned}$$

and

$$\int_{\Delta_1^M} g^*(\xi)^p d\omega_{z_0}^M(\xi) = \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{-p/q}\omega_{z_0}^M(\{\zeta_{n_l}\})$$
$$= \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{1-p/q}$$
$$\leq \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{(q-p)/q}$$
$$\leq \sum_{l=1}^{\infty} 1/l^2 < \infty.$$

By Proposition 3 we find that $g \in h_p(R) \setminus h_q(R)$. By (i) this is a contradiction. Hence, there exists a nullset N of Δ^M with respect to $\omega_{z_0}^M$ such that $\Delta_1^M \setminus N$ consists of finitely many points with positive harmonic measure, and so, we get (ii).

Suppose that (ii) holds. Fix a point z_0 of R. We can find a nullset N of Δ^M with respect to $\omega_{z_0}^M$ such that $\Delta_1^M \setminus N$ consists of at finitely many points with positive harmonic measure. Let n_0 be the cardinal number of $\Delta_1^M \setminus N$. Set $\Delta_1^M \setminus N = \{\zeta_n\}_{n=1}^{n_0}$. Let p and q be real numbers with $p < q$ and $p > 1$. Clearly $h_q(R) \subset h_p(R)$. Take any element of h of $h_p(R)$. By definition of $h_p(R)$ h has a minimal fine limit $h^*(\xi)$ at almost every point $\xi \in \Delta_1^M$ with respect to $\omega_{z_0}^M$ such that $h(z) = \int_{\Delta_1^M} h^*(\xi) d\omega_z^M(\xi)$ and $\int_{\Delta_1^M} |h^*(\xi)|^p d\omega_{z_0}^M(\xi) < \infty$. By (ii) we have

$$h(z) = \int_{\Delta_1^M} h^*(\xi) d\omega_z^M(\xi) = \sum_{n=1}^{n_0} h^*(\zeta_n)\omega_z^M(\{\zeta_n\})$$

and

$$\int_{\Delta_1^M} |h^*(\xi)|^p d\omega_{z_0}^M(\xi) = \sum_{n=1}^{n_0} |h^*(\zeta_n)|^p \omega_{z_0}^M(\{\zeta_n\}) < \infty.$$

Hence $\dim h_p(R) \leq n_0$ and $|h^*(\zeta_n)| < \infty$ $(n = 1, \ldots, n_0)$. Thus,

$$\int_{\Delta_1^M} |h^*(\xi)|^q d\omega_{z_0}^M(\xi) = \sum_{n=1}^{n_0} |h^*(\zeta_n)|^q \omega_{z_0}^M(\{\zeta_n\}) < \infty,$$

and hence $h \in h_q(R)$, that is, $h_p(R) \subset h_q(R)$. Hence $h_p(R) = h_q(R)$. Hence $\dim h_q(R) = \dim h_p(R) \leq n_0 < \infty$. We get (iii).

Suppose that (iii) holds. Let p and q be real numbers with $p < q$ and $p > 1$. Since $h_p(R)$ and $h_q(R)$ are linear spaces and $h_q(R)$ is a subspace of $h_p(R)$, by (iii), we find that $h_p(R) = h_q(R)$. Hence we get (i).

Next we consider $q = \infty$. Suppose that (i) holds. Take a real number p' with $p' > p$. Then $HB(R) \subset h_{p'}(R) \subset h_p(R)$. By (i) $h_{p'}(R) = h_p(R)$. By the implication: (i) $\Rightarrow$ (ii) in the case that $q \neq \infty$ we get (ii).

Suppose that (ii) holds. Fix a point z_0 of R. We can find a nullset N of Δ^M with respect to $\omega_{z_0}^M$ such that $\Delta_1^M \setminus N$ consists of finitely many points with positive harmonic measure. Let n_0 be the cardinal number of $\Delta_1^M \setminus N$. Set $\Delta_1^M \setminus N = \{\zeta_n\}_{n=1}^{n_0}$. Clearly $HB(R) \subset h_p(R)$. Take any element h of

$h_p(R)$. By definition of $h_p(R)$ we find that $h(z) = \int_{\Delta_1^M} h^*(\xi) d\omega_z^M(\xi)$ and $\int_{\Delta_1^M} |h^*(\xi)|^p d\omega_{z_0}^M(\xi) < \infty$. By (ii) we have

$$h(z) = \int_{\Delta_1^M} h^*(\xi) d\omega_z^M(\xi) = \sum_{n=1}^{n_0} h^*(\zeta_n)\omega_z^M(\{\zeta_n\})$$

and

$$\int_{\Delta_1^M} |h^*(\xi)|^p d\omega_{z_0}^M(\xi) = \sum_{n=1}^{n_0} |h^*(\zeta_n)|^p \omega_{z_0}^M(\{\zeta_n\}) < \infty.$$

Hence $\dim h_p(R) \leq n_0$ and $|h^*(\zeta_n)| < \infty$ $(n = 1, \ldots, n_0)$. Thus, $h \in HB(R)$, that is, $h_p(R) \subset HB(R)$. Hence $h_p(R) = HB(R)$. Hence $\dim HB(R) = \dim h_p(R) \leq n_0 < \infty$. We get (iii).

Suppose that (iii) holds. Since $h_p(R)$ and $HB(R)$ are linear spaces and $HB(R)$ is a subspace of $h_p(R)$, by (iii), we find that $h_p(R) = HB(R)$. Hence we get (i).

Therefore we have the desired result.

3.2 Proof of Theorem 2

First we consider $q \neq \infty$. Suppose that (i) holds, that is, $HP(R) = h_q(R)$ $(q > 1)$. Let h be a minimal harmonic function on R. Clearly $h \in HP_+(R)$. By (i) $h \in HP_+(R) \cap h_q(R) = h_{q+}(R)$. Since $h_{q+}(R) \subset MHB_+(R)$, $h \in MHB_+(R)$. Thus there exists a monotone increasing sequence $\{h_n\}_{n=1}^\infty$ of $HB_+(R)$ such that $h_n \neq 0$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} h_n = h$ on R. By minimality of h there exists a positive constant α such that $h = \alpha h_1$ on R. Hence h is bounded on R. Let ζ_h be the element of Δ_1^M coressponding to h. Fix a point z_0 of R. Since h is minimal, there exists a positive constant β with $h = \beta k_{\zeta_h}$. Hence, because h is bounded on R, by Proposition 1 we find that the harmonic measure $\omega_{z_0}^M(\{\zeta_h\})$ of $\{\zeta_h\}$ is positive. Hence, Δ_1^M consists of at most countably many points with positive harmonic measure.

Suppose that $\sharp\Delta_1^M = \aleph_0$. Set $\Delta_1^M = \{\zeta_n\}_{n=1}^\infty$. Hence there exists a subsequence $\{\zeta_{n_l}\}_{l=1}^\infty$ of $\{\zeta_n\}_{n=1}^\infty$ with $\omega_{z_0}^M(\{\zeta_{n_l}\}) \leq 1/l^{2q/(q-1)}$ $(l \in \mathbb{N})$. Set

$$g^*(\xi) = \begin{cases} [\omega_{z_0}^M(\{\zeta_{n_l}\})]^{-1/q}, & \text{for } \xi = \zeta_{n_l}, \\ 0, & \text{for } \xi \in \Delta^M \setminus \{\zeta_{n_l}\}_{l=1}^\infty. \end{cases}$$

And set $g(z) = \int g^*(\xi) d\omega_z^M(\xi)$. Then we have

$$\int g^*(\xi)^q d\omega_{z_0}^M(\xi) = \sum_{l=1}^\infty [\omega_{z_0}^M(\{\zeta_{n_l}\})]^{-1} \omega_{z_0}^M(\{\zeta_{n_l}\})$$
$$= \sum_{n=1}^\infty 1 = \infty,$$

and

$$\begin{aligned}\int g^*(\xi)d\omega_{z_0}^M(\xi) &= \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{-1/q}\omega_{z_0}^M(\{\zeta_{n_l}\}) \\ &= \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{1-1/q} \\ &\le \sum_{l=1}^{\infty}[\omega_{z_0}^M(\{\zeta_{n_l}\})]^{(q-1)/q} \\ &\le \sum_{l=1}^{\infty} 1/l^2 < \infty.\end{aligned}$$

By Proposition 3 we find that $g \in MHB(R) \setminus h_q(R)$. Since $HP(R) \supset MHB(R)$, by (i), this is a contradiction. Hence Δ_1^M consists of finitely many points with positive harmonic measure, and so, we get (ii).

Suppose that (ii) holds. Let n_0 be the cardinal number of Δ_1^M. Set $\Delta_1^M = \{\zeta_n\}_{n=1}^{n_0}$. Let $q > 1$. Clearly $h_q(R) \subset HP(R)$. Take any element h of $HP(R)$. By Proposition 2 we find a signed measure μ on Δ^M that $\mu(\Delta^M \setminus \Delta_1^M) = 0$ and $h(z) = \int_{\Delta_1^M} \omega_z^M(\{\xi\})d\mu(\xi)$. By (ii) we have

$$h(z) = \int_{\Delta_1^M} \omega_z^M(\{\xi\})d\mu(\xi) = \sum_{n=1}^{n_0} \omega_z^M(\{\zeta_n\})\mu(\{\zeta_n\}) = \int_{\Delta_1^M} \mu(\{\xi\})d\omega_z^M(\xi)$$

and

$$|\mu(\{\zeta_n\})| < \infty \ (n = 1, \dots, n_0).$$

Fix a point z_0 of R. We have

$$\int_{\Delta_1^M} |\mu(\{\xi\})|^q d\omega_{z_0}^M(\xi) = \sum_{n=1}^{n_0} |\mu(\{\zeta_n\})|^q \omega_{z_0}^M(\{\zeta_n\}) < \infty.$$

Hence $\dim HP(R) \le n_0$ and by Proposition 3, $h \in h_q(R)$, that is, $HP(R) \subset h_q(R)$. Hence $HP(R) = h_q(R)$. Hence $\dim h_q(R) = \dim HP(R) \le n_0 < \infty$. We get (iii).

Suppose that (iii) holds. Let $q > 1$. Since $HP(R)$ and $h_q(R)$ are linear spaces and $h_q(R)$ is a subspace of $HP(R)$, by (iii), we find that $HP(R) = h_q(R)$. Hence we get (i).

Next we consider $q = \infty$. Suppose that (i) holds, that is, $HB(R) = HP(R)$. Let $q' > 1$. Since $HB(R) \subset h_{q'}(R) \subset HP(R)$, $h_{q'}(R) = HP(R)$. By the implication: (i) $\Rightarrow$ (ii) in the case that $q \neq \infty$ we get (ii).

Suppose that (ii) holds. Let n_0 be the cardinal number of Δ_1^M. Set $\Delta_1^M = \{\zeta_n\}_{n=1}^{n_0}$. Clearly $HB(R) \subset HP(R)$. Take any element h of $HP(R)$. By Proposition 2 we find a signed measure μ on Δ^M such that $\mu(\Delta^M \setminus \Delta_1^M) = 0$ and $h(z) = \int_{\Delta_1^M} \omega_z^M(\{\xi\})d\mu(\xi)$. By (ii) we have

$$h(z) = \int_{\Delta_1^M} \omega_z^M(\{\xi\})d\mu(\xi) = \sum_{n=1}^{n_0} \mu(\{\zeta_n\})\omega_z^M(\{\zeta_n\}).$$

Hence $\dim HP(R) \le n_0$ and $h \in HB(R)$, that is, $HP(R) \subset HB(R)$. Hence $HP(R) = HB(R)$. Hence $\dim HB(R) = \dim HP(R) \le n_0 < \infty$. We get (iii).

Suppose that (iii) holds. Since $HP(R)$ and $HB(R)$ are linear spaces and $HB(R)$ is a subspace of $HP(R)$, by (iii), we find that $HP(R) = HB(R)$. Hence we get (i).

Therefore we have the desired result.

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いつ双曲的リーマン面上の異なる指数をもつ調和ハーディ空間は同一の集合になるか?

——還暦を祝して石田久教授に捧げる——

正　岡　弘　照

要　旨

Rを双曲的（グリーン関数が存在する）リーマン面とする。$1 \leq p < q \leq \infty$ を仮定する。この論文では, 調和ハーディ空間 $h_p(R)$ と $h_q(R)$ が同一の集合であるための特徴づけをRのマルチン境界 Δ^M の言葉で与える。Δ_1^M をRのミニマルマルチン境界とする。$p > 1$ の場合, $h_p(R)$ と $h_q(R)$ が同一の集合であるための必要十分条件はΔ^M の部分集合 N が 存在して, その Δ^M 上の調和測度は 0 で, $\Delta_1^M \setminus N$ が 有限個のΔ^M 上の調和測度が正の点からなることである。$p = 1$ である場合, $h_1(R)$ と $h_q(R)$ が同一の集合であるための必要十分条件はΔ_1^M が有限個のΔ^M 上の調和測度が正の点からなることである。

キーワード：双曲的リーマン面, 調和ハーディ空間, マルチン境界, ミニマルマルチン境界, 調和測度