

An estimation of Hausdorff dimension of totally disconnected Julia sets of $z^n + c$

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(Received September 30, 2008,
Revised December 12, 2008)

Abstract

Let $f_c(z) = z^n + c$, where n is a positive integer greater than one and c is a complex number. For sufficiently large $|c|$, the Julia set of $J(f_c)$ of $f_c(z)$ is totally disconnected and the Hausdorff dimension $\dim_H J(f_c)$ of $J(f_c)$ satisfies

$$\frac{n \log n}{(n-1) \log(n^{\frac{n}{n-1}}(|c| + |2c|^{\frac{1}{n}}))} \leq \dim_H J(f_c) \leq \frac{n \log n}{(n-1) \log(n^{\frac{n}{n-1}}(|c| - |2c|^{\frac{1}{n}}))}.$$

Keywords: polynomial, totally disconnected set, Julia set, Hausdorff dimension, convex set

1. Introduction

Let $f_c(z) = z^n + c$, where n is a positive integer greater than one and c is a complex number. We denote the Julia set of f_c by $J(f_c)$ and the Hausdorff dimension of $J(f_c)$ by $\dim_H J(f_c)$.

Falconer([1]) proved that $J(f_c)$ is totally disconnected and

$$\frac{2 \log 2}{\log(4(|c| + \sqrt{|2c|}))} \leq \dim_H J(f_c) \leq \frac{2 \log 2}{\log(4(|c| - \sqrt{|2c|}))}$$

when $n = 2$ and $|c| > (5 + 2\sqrt{6})/4$.

L. Fang and C. Zhang ([2]) gave an estimate of the Hausdorff dimension of $J(f_c)$ in case that $n = 2, 3$. In this note we shall give an estimation of the Hausdorff dimension of Julia sets $J(f_c)$ for $n \geq 2$.

We note that there is a unique positive number $\rho(n)$ satisfying $\rho(n) - (2\rho(n))^{\frac{1}{n}} - n^{\frac{n}{1-n}} = 0$. We note that $\rho(n) > 2^{\frac{1}{n-1}}$ and $\rho(2) = (5 + 2\sqrt{6})/4$.

Main Theorem. Suppose $f_c(z) = z^n + c$ and $|c| > \rho(n)$. Then $J(f_c)$ is totally disconnected and

$$\frac{n \log n}{(n-1) \log(n^{\frac{n}{n-1}}(|c| + |2c|^{\frac{1}{n}}))} \leq \dim_H J(f_c) \leq \frac{n \log n}{(n-1) \log(n^{\frac{n}{n-1}}(|c| - |2c|^{\frac{1}{n}}))}.$$

We shall prove Main Theorem in Section 3. And we give an alternative proof of Main Theorem in Section 4.

2. Preliminaries

Lemma 1. *It holds that for $1 < j < n$*

$$\operatorname{Re}((\rho_1 e^{i\varphi} + a)^{\frac{j}{n}} (\rho_2 e^{i\psi} + a)^{\frac{n-1-j}{n}}) \geq (a-1)^{\frac{n-1}{n}}$$

where $a > 1$, $0 \leq \rho_1, \rho_2 \leq 1$, φ, ψ are real numbers and $|\arg(z + a)^{\frac{1}{n}}| < \pi/n$ for $|z| \leq 1$.

Proof. Let $\zeta_1 = \rho_1 e^{i\varphi} + a$, $\zeta_2 = \rho_2 e^{i\psi} + a$ and $K = \{|z - a| \leq 1\}$. We shall prove that for any $\zeta_1, \zeta_2 \in K$ there is a point $\zeta^* \in K$ such that $\zeta_1^j \zeta_2^{n-1-j} = (\zeta^*)^{n-1}$. Let $\operatorname{Log} z$ be the principal branch of $\log z$ on K . Since the image $\operatorname{Log}(K)$ of K by $\operatorname{Log} z$ is convex,

$$\frac{j \operatorname{Log} \zeta_1 + (n-1-j) \operatorname{Log} \zeta_2}{n-1} \in \operatorname{Log}(K).$$

Hence,

$$\zeta^* = \exp\left(\frac{j \operatorname{Log} \zeta_1 + (n-1-j) \operatorname{Log} \zeta_2}{n-1}\right) \in K$$

and $\zeta_1^j \zeta_2^{n-1-j} = (\zeta^*)^{n-1}$.

Thus

$$\begin{aligned} \operatorname{Re}((\rho_1 e^{i\varphi} + a)^{\frac{j}{n}} (\rho_2 e^{i\psi} + a)^{\frac{n-1-j}{n}}) &\geq \min_{0 \leq r \leq 1, 0 \leq \varphi \leq 2\pi} \operatorname{Re}((re^{i\varphi} + a)^{\frac{n-1}{n}}) \\ &= (a-1)^{\frac{n-1}{n}}. \end{aligned}$$

□

Lemma 2. *Consider $F(z) = (z + a)^{\frac{1}{n}}$ on $\{|z| < 1\}$, where $a > 1$ and the branch is chosen so that $F(0) = a^{\frac{1}{n}} > 1$. Then $F(\{|z| < 1\})$ is convex.*

Proof. We note that $g(z) = (F(z) - F(0))/F'(0)$ is holomorphic and univalent on $\{|z| < 1\}$ and $g(0) = 0$, $g'(0) = 1$. By a simple calculation, we have

$$z \frac{g''(z)}{g'(z)} = \frac{1-n}{n} \frac{z}{z+a}.$$

Hence,

$$\operatorname{Re}\left(z \frac{g''(z)}{g'(z)}\right) > -\frac{n-1}{n} \frac{1}{a+1} > -1.$$

Thus, $F(\{|z| < 1\})$ is convex. □

Falconer proved the followings (Theorem 9.1, Proposition 9.6 and Proposition 9.7 in [1]). We quote the theorems without proofs.

Proposition 3. *Consider the iterated function system (IFS) by the contractions $\{S_1, S_2, \dots, S_n\}$ on a closed subset D of the complex plane \mathbb{C} , so that*

$$|S_j(z_1) - S_j(z_2)| \leq c_j |z_1 - z_2| \quad (z_1, z_2 \in D)$$

with $0 < c_j < 1$ ($j = 1, 2, \dots, n$). Then there is a unique attractor F , i.e. a non-empty compact set such that

$$F = \sum_{j=1}^n S_j(F).$$

Proposition 4. Let F be the attractor of an IFS consisting of contractions $\{S_1, S_2, \dots, S_n\}$ on a closed subset D of the complex plane \mathbb{C} such that

$$b_j|z_1 - z_2| \leq |S_j(z_1) - S_j(z_2)| \leq c_j|z_1 - z_2| \quad (z_1, z_2 \in D)$$

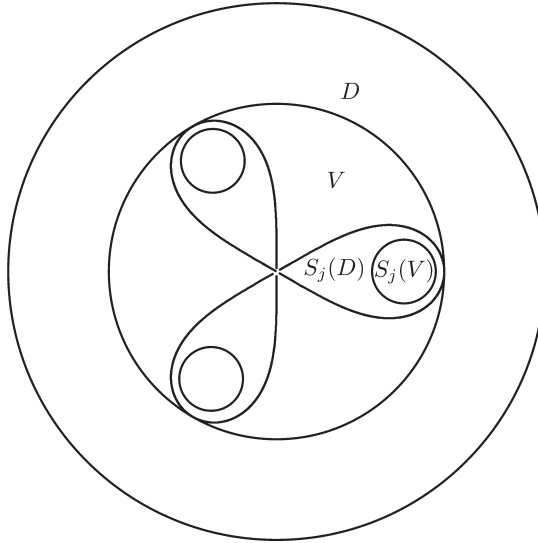
with $0 < b_j < 1$ and $0 < c_j < 1$ ($j = 1, 2, \dots, n$). Then F is totally disconnected and $s \leq \dim_H F \leq t$, where $\sum_{j=1}^n b_j^s = 1$, $\sum_{j=1}^n c_j^t = 1$.

3. Proof of Main Theorem

Let C be the circle $\{|z| = |c|\}$ and D its interior $\{|z| < |c|\}$. Then $f_c^{-1}(C)$ is a figure of mutually disjoint n -loops with self-intersection point at 0. Since $|c| > \rho(n) > 2^{\frac{1}{n-1}}$, we have $f_c^{-1}(C) \subset D$.

The interior of each of the loops of $f_c^{-1}(C)$ is mapped by f_c in a bijective manner onto D . If we define S_j ($j = 1, \dots, n$) as the branches of f_c^{-1} inside each loop, then $S_j(D)$ ($j = 1, \dots, n$) are the interiors of the n loops.

Let $V = \{|z| < |2c|^{\frac{1}{n}}\}$. Since $|2c|^{\frac{1}{n}} < |c|$, we have $\overline{S_j(V)} \subset S_j(D) \subset V \subset \overline{V} \subset D$ ($j = 1, \dots, n$).



Hence, for any $z_1, z_2 \in \overline{V}$ and each j ($j = 1, \dots, n$), we have

$$\begin{aligned} |S_j(z_1) - S_j(z_2)| &= |(z_1 - c)^{\frac{1}{n}} - (z_2 - c)^{\frac{1}{n}}| \\ &= \frac{|z_1 - z_2|}{|(z_1 - c)^{\frac{n-1}{n}} + (z_1 - c)^{\frac{n-2}{n}}(z_2 - c)^{\frac{1}{n}} + \dots + (z_2 - c)^{\frac{n-1}{n}}|}, \end{aligned}$$

where we write $S_j(z) = (z - c)^{\frac{1}{n}}$.

Here,

$$\begin{aligned} & |(z_1 - c)^{\frac{n-1}{n}} + (z_1 - c)^{\frac{n-2}{n}}(z_2 - c)^{\frac{1}{n}} + \cdots + (z_2 - c)^{\frac{n-1}{n}}| \\ & \leq |z_1 - c|^{\frac{n-1}{n}} + |z_1 - c|^{\frac{n-2}{n}}|z_2 - c|^{\frac{1}{n}} + \cdots + |z_2 - c|^{\frac{n-1}{n}} \\ & \leq n(|2c|^{\frac{1}{n}} + |c|)^{\frac{n-1}{n}}. \end{aligned}$$

To get a lower bound, let $z_1 = \rho_1|2c|^{\frac{1}{n}}e^{i\alpha}$, $z_2 = \rho_2|2c|^{\frac{1}{n}}e^{i\beta}$ and $c = |2c|^{\frac{1}{n}}c_0$, where $0 \leq \rho_1, \rho_2 \leq 1$ and $0 \leq \alpha, \beta \leq 2\pi$. Note that $|c_0| > 1$. Moreover, let $\alpha' = \alpha - \arg c_0 - \pi$ and $\beta' = \beta - \arg c_0 - \pi$. Then

$$\begin{aligned} & |(z_1 - c)^{\frac{n-1}{n}} + (z_1 - c)^{\frac{n-2}{n}}(z_2 - c)^{\frac{1}{n}} + \cdots + (z_2 - c)^{\frac{n-1}{n}}| \\ & = |2c|^{\frac{n-1}{n^2}} |(\rho_1 e^{i\alpha} - c_0)^{\frac{n-1}{n}} + (\rho_1 e^{i\alpha} - c_0)^{\frac{n-2}{n}}(\rho_2 e^{i\beta} - c_0)^{\frac{1}{n}} + \cdots + (\rho_2 e^{i\beta} - c_0)^{\frac{n-1}{n}}| \\ & = |2c|^{\frac{n-1}{n^2}} |(\rho_1 e^{i\alpha'} + |c_0|)^{\frac{n-1}{n}} + (\rho_1 e^{i\alpha'} + |c_0|)^{\frac{n-2}{n}}(\rho_2 e^{i\beta'} + |c_0|)^{\frac{1}{n}} + \cdots + (\rho_2 e^{i\beta'} + |c_0|)^{\frac{n-1}{n}}| \\ & \geq |2c|^{\frac{n-1}{n^2}} \operatorname{Re}\{(\rho_1 e^{i\alpha'} + |c_0|)^{\frac{n-1}{n}} + (\rho_1 e^{i\alpha'} + |c_0|)^{\frac{n-2}{n}}(\rho_2 e^{i\beta'} + |c_0|)^{\frac{1}{n}} + \cdots + (\rho_2 e^{i\beta'} + |c_0|)^{\frac{n-1}{n}}\}. \end{aligned}$$

Here, note that we choose a branch of $(z + |c_0|)^{\frac{1}{n}}$ so that $|\arg(z + |c_0|)^{\frac{1}{n}}| < \pi/n$ for $|z| \leq 1$. Clearly

$$\operatorname{Re}(\rho_1 e^{i\alpha'} + |c_0|)^{\frac{n-1}{n}}, \quad \operatorname{Re}(\rho_2 e^{i\beta'} + |c_0|)^{\frac{n-1}{n}} \geq (|c_0| - 1)^{\frac{n-1}{n}}.$$

By Lemma 1, we have for $0 < j < n$

$$\operatorname{Re}((\rho_1 e^{i\alpha'} + |c_0|)^{\frac{j}{n}}(\rho_2 e^{i\beta'} + |c_0|)^{\frac{n-1-j}{n}}) \geq (|c_0| - 1)^{\frac{n-1}{n}}.$$

Thus we have

$$\begin{aligned} & |(z_1 - c)^{\frac{n-1}{n}} + (z_1 - c)^{\frac{n-2}{n}}(z_2 - c)^{\frac{1}{n}} + \cdots + (z_2 - c)^{\frac{n-1}{n}}| \geq n|2c|^{\frac{n-1}{n^2}}(|c_0| - 1)^{\frac{n-1}{n}} \\ & = n|2c|^{\frac{n-1}{n^2}} \left(\frac{|c| - |2c|^{\frac{1}{n}}}{|2c|^{\frac{1}{n}}} \right)^{\frac{n-1}{n}} = n(|c| - |2c|^{\frac{1}{n}})^{\frac{n-1}{n}}. \end{aligned}$$

Therefore, we get

$$\frac{1}{n} \left(|c| + |2c|^{\frac{1}{n}} \right)^{-\frac{n-1}{n}} \leq \frac{|S_i(z_1) - S_i(z_2)|}{|z_1 - z_2|} \leq \frac{1}{n} \left(|c| - |2c|^{\frac{1}{n}} \right)^{-\frac{n-1}{n}}.$$

The upper bound is less than 1 if $|c| > \rho(n)$, in which case S_1, S_2, \dots, S_n are contractions on the disk \bar{V} . By Proposition 3, there is a unique non-empty compact attractor $F \subset \bar{V}$ satisfying $F = \sum_{j=1}^n S_j(F)$. Since $\{S_1(\bar{V}), S_2(\bar{V}), \dots, S_n(\bar{V})\}$ are disjoint, so are $\{S_1(F), S_2(F), \dots, S_n(F)\}$, implying F is totally disconnected.

We know that F coincides with $J(f_c)$ (see [1, p.229]).

By Proposition 4,

$$n \left(\frac{1}{n} (|c| - |2c|^{\frac{1}{n}})^{-\frac{n-1}{n}} \right)^t = 1$$

and

$$n \left(\frac{1}{n} (|c| + |2c|^{\frac{1}{n}})^{-\frac{n-1}{n}} \right)^s = 1$$

imply the estimation of Theorem 1. \square

4. Alternative proof of Main Theorem

We can estimate $|S_j(z_1) - S_j(z_2)|/|z_1 - z_2|$ by using differentiation.

Since V and $S_j(V)$ are convex by Lemma 2,

$$\inf_{z \in V} |S'_j(z)| \leq \frac{|S_j(z_1) - S_j(z_2)|}{|z_1 - z_2|} \leq \sup_{z \in V} |S'_j(z)|.$$

Here,

$$\sup_{z \in V} |S'_j(z)| = \frac{1}{n} \sup_{z \in V} |(z - c)|^{\frac{1-n}{n}} \leq \frac{1}{n} (|c| - |2c|^{\frac{1}{n}})^{\frac{1-n}{n}},$$

and

$$\inf_{z \in V} |S'_j(z)| = \frac{1}{n} \inf_{z \in V} |(z - c)|^{\frac{1-n}{n}} \geq \frac{1}{n} (|c| + |2c|^{\frac{1}{n}})^{\frac{1-n}{n}}.$$

Therefore, we get

$$\frac{1}{n} \left(|c| + |2c|^{\frac{1}{n}} \right)^{-\frac{n-1}{n}} \leq \frac{|S_j(z_1) - S_j(z_2)|}{|z_1 - z_2|} \leq \frac{1}{n} \left(|c| - |2c|^{\frac{1}{n}} \right)^{-\frac{n-1}{n}}.$$

Repeating the same discussion as in Section 3, we have the desired result. \square

References

- [1] Falconer, K. J., Fractal geometry (2nd ed.). Wiley, 2003.
- [2] Fang, L. and Zhang, C., Better Hausdorff dimension estimation of quadratic and cubic functions' Julia sets. J. Beijing Inst. Tech. 15 (2006), 123–126.

$z^n + c$ の全不連結なジュリア集合のハウスドルフ次元の評価

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要 旨

$f_c(z) = z^n + c$ とする．ここで， n は 2 以上の整数， c は複素数である． $|c|$ が十分大きいとき， $f_c(z)$ のジュリア集合 $J(f_c)$ は全不連結であり，そのハウスドルフ次元 $\dim_H J(f_c)$ について次の評価が成り立つ．

$$\frac{n \log n}{(n-1) \log \left(n^{\frac{n}{n-1}} (|c| + |2c|^{\frac{1}{n}}) \right)} \leq \dim_H J(f_c) \leq \frac{n \log n}{(n-1) \log \left(n^{\frac{n}{n-1}} (|c| - |2c|^{\frac{1}{n}}) \right)}.$$

キーワード：多項式，全不連結集合，ジュリア集合，ハウスドルフ次元，凸集合