Quantile hedging and minimizing the expected shortfall for bond options

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Abstract

Föllmer and Leukert ([2, 3]) investigated the quantile hedging problem and the shortfall risk hedging problem for options of stocks. In the first part of this paper, the quantile hedging problems for options of bonds and for caplets are formulated. We derive a formula of the success probability for Ho-Lee bond model with the market price of risk γ_t which is equal to $\sigma t + c$ where σ is the volatility and c is a constant. Furthermore a lower bound of the success probability is calculated for caplets with the market price of risk γ_t which is equal to $\sigma t + c$.

In the second part of this paper the shortfall risk problems for options of bonds and for caplets are formulated. We give a formula of the minimal expected shortfall for Ho-Lee bond model with the constant market price of risk.

Keywords: quantile hedging, minimizing the expected shortfall, bond options, caplets, Ho-Lee bond model

1. Preliminaries

In this section we summarize the results of Föllmer and Leukert ([2, 3]), that is, quantile hedging and optimal partial hedging with the expected shortfall for stocks.

For the terminology, see the papers of Föllmer and Leukert ([2, 3]). Assume that a market is arbitrage-free and complete. Let B_t be the cash bond. Let X_t be a discounted stock price which is a semimartingale under $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\{\mathcal{F}_t\}_{\in[0,T]}$.

A self-financing strategy is defined by an initial capital V_0 and by a predictable process ξ which is an itegrand for the semimartingale $\{X_t\}$. A self-financing strategy (V_0, ξ) is called admissible if the process V defined by

$$V_t = V_0 + \int_0^t \xi_s dX_s \quad t \in [0, T], \quad \mathbf{P} - a.s.$$

satisfies $V_t \geq 0$ for all $t \in [0, T]$, $\mathbf{P}$ − a.e..

Let $\mathbf{P}^*$ be the unique equivalent martingale measure. Let H be a discounted non-negative contingent claim at a claim time-horizon T; $H \in L^1(\mathbf{P}^*), \mathcal{F}_T$−measurable. Completeness implies that there exists a perfect hedge, i.e. a predictable process ξ such that

$$\mathbf{E}_{\mathbf{P}^*}[H|\mathcal{F}_t] = H_0 + \int_0^t \xi_s dX_s \quad \text{for all} \quad t \in [0, T], \quad \mathbf{P} - \text{a.e.}$$

Let $H_0 = \mathbf{E}_{\mathbf{P}^*}[H]$ which is the worth of the contingent claim at time 0.

A quantile hedging problem with the initial cost $\tilde{V}_0$ is as follows:

Fix the initial cost $\tilde{V}_0(< H_0)$. Find an admissible strategy (V_0, ξ) such that

$$\mathbf{P}[V_0 + \int_0^T \xi_s dX_s \geq H] = \max \tag{$*$}$$

under the constraint $V_0 \leq \tilde{V}_0$.

The set $\{V_0 + \int_0^T \xi_s dX_s \geq H\}$ is called a success set. Let $\mathbf{Q}^*$ be a probability measure such that

$$\frac{\mathbf{dQ}^*}{\mathbf{dP}^*} = \frac{H}{H_0}.$$

Theorem 1 (Föllmer & Leukert ([2]))**.** Assume that $\tilde{A} = \{\frac{\mathbf{dP}}{\mathbf{dP}^*} > aH\}$ satisfies $\mathbf{Q}^*[\tilde{A}] = \frac{\tilde{V}_0}{H_0}$. Let $(\tilde{V}_0, \tilde{\xi})$ be the perfect hedge for the knockout option $HI_{\tilde{A}}$ where $I_{\tilde{A}}(\omega) = 1$ if $\omega \in \tilde{A}$, and $= 0$ if $\omega \notin \tilde{A}$.

Then $(\tilde{V}_0, \tilde{\xi})$ solves the optimal Problem $(*)$. Furthermore the success set for $(\tilde{V}_0, \tilde{\xi})$ is equal to $\tilde{A} \quad \mathbf{P} - a.e.$

An optimal partial hedging problem with the expected shortfall with the initial cost $\tilde{V}_0$ is as follows:

Fix the initial cost $\tilde{V}_0(< H_0)$. Find an admissible strategy (V_0, ξ) such that

$$\mathbf{E}[(H - V_T)^+] = \min \tag{$**$}$$

under the constraint $V_0 \leq \tilde{V}_0$ where $V_T = V_0 + \int_0^T \xi_s dX_s$

The excess $(H - V_T)^+$ is called a shortfall.

Let $\mathbf{Q}$ and $\mathbf{Q}^*$ be probability measures such that

$$\frac{\mathbf{dQ}}{\mathbf{dP}} = \frac{H}{\mathbf{E}[H]} \quad \text{and} \quad \frac{\mathbf{dQ}^*}{\mathbf{dP}^*} = \frac{H}{H_0}.$$

Applying the Neyman Pearson lemma to $\mathbf{Q}$ and $\mathbf{Q}^*$, we obtain the following theorem.

Theorem 2 (Föllmer & Leukert ([3]))**.** Assume that $\tilde{A} = \{\frac{\mathbf{dP}}{\mathbf{dP}^*} > a\}$ satisfies $\mathbf{Q}^*[\tilde{A}] = \frac{\tilde{V}_0}{H_0}$. Let $(\tilde{V}_0, \tilde{\xi})$ be the the perfect hedge for the knockout option $HI_{\tilde{A}}$. Then $(\tilde{V}_0, \tilde{\xi})$ solves the optimal Problem $(*)$.

2. Quantile hedging for bond options

In this section the quantile hedging problems for options of bonds are formulated. We derive an explicit formula of the success probability for Ho-Lee bond model with the market price of risk γ_t which is equal to $\sigma t + c$ where c is a constant. First we summarize the facts of the Ho-Lee bond model for bonds with the market price of risk γ_t. See Baxter [1].

The Ho-Lee bond model with the market price of risk γ_t under P
Fact 1. the forward rate

$$df(t,T) = \sigma dW_t + (\sigma^2(T-t) + \sigma\gamma_t)dt$$

Fact 2. the bond prices

$$P(t,T) = \exp\left(-\left\{\sigma(T-t)W_t + \int_t^T f(0,u)du + \frac{\sigma^2}{2}T(T-t)t + \sigma(T-t)\int_0^t \gamma_s ds\right\}\right)$$

Fact 3. the cash bond

$$B_t = \exp\left(\sigma\int_0^t W_s ds + \int_0^t f(0,u)du + \frac{1}{6}\sigma^2 t^3 + \int_0^t \sigma(t-s)\gamma_s ds\right)$$

Fact 4. the derivative $\frac{\mathbf{dP^*}}{\mathbf{dP}}$

$$\frac{\mathbf{dP^*}}{\mathbf{dP}} = \exp\left(-\int_0^t \gamma_s dW_s - \int_0^t \frac{\gamma_s^2}{2} ds\right)$$

where $\mathbf{P}^*$ is the unique equivalent martingale measure.

A discounted option $H_t = (P(t,T)-k)^+/B_t$ on T-bond (i.e., with maturity T), struck at k with exercise time t, is worth

$$H_0 = \mathbf{E}_{\mathbf{P}^*}[H] = \mathbf{E}_{\mathbf{P}^*}[B_t^{-1}(P(t,T)-k)^+]$$

at time 0.

Note that

$$P(t,T) = \exp\left(-\left\{\sigma(T-t)W_t^* + \int_t^T f(0,u)du + \frac{\sigma^2}{2}T(T-t)t\right\}\right)$$

$$B_t = \exp\left(\sigma\int_0^t W_s^* ds + \int_0^t f(0,u)du + \frac{1}{6}\sigma^2 t^3\right)$$

where $W_t^* = W_t + \int_0^t \gamma_s ds$ which is a P^*-Brownian motion. Then H_0 is evaluated (cf. [1]) by

$$H_0 = P(0,t)\left\{F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}\right)\right\}$$

where

$$F = \frac{P(0,T)}{P(0,t)} = \frac{\exp\left(-\int_0^T f(0,u)du\right)}{\exp\left(-\int_0^t f(0,u)du\right)}$$

and Φ is the distribution function of the standard normal distribution.

Fix $\tilde{V}_0(< H_0)$. A quantile hedging of the above bond call option with the initial cost $\tilde{V}_0$ is calculated as follows (cf. Section 1 and [2]):

The success set A is

$$A = \{\frac{\mathbf{dP}}{\mathbf{dP^*}} > aH\} = \{B_t\frac{\mathbf{dP}}{\mathbf{dP^*}} > a(P(t,T)-k)^+\}$$

where

$$
\begin{aligned}
B_t \frac{\mathbf{dP}}{\mathbf{dP}^*} &= \exp(\sigma \int_0^t W_s ds + \int_0^t \gamma_s dW_s + a_1(t)) \\
P(t,T) &= \exp(-\sigma(T-t)W_t + a_2(t)) \\
a_1(t) &= \int_0^t f(0,u)du + \frac{1}{6}\sigma^2 t^3 + \int_0^t \sigma(t-s)\gamma_s ds + \int_0^t \frac{\gamma_s^2}{2} ds
\end{aligned}
$$

and

$$
a_2(t) = -(\int_t^T f(0,u)du + \frac{\sigma^2}{2} T(T-t)t + \sigma(T-t)\int_0^t \gamma_s ds).
$$

For further investigation, we assume that

$$
\gamma_s = \sigma s,
$$

that is, the market price of risk γ_s is propotional to time s with the volatility σ.

Then

$$
\sigma \int_0^t W_s ds + \int_0^t \gamma_s ds = \sigma t W_t
$$

and so

$$
B_t \frac{\mathbf{dP}}{\mathbf{dP}^*} = \exp(\sigma t W_t + a_1(t))
$$

and

$$
\begin{aligned}
P(t,T) &= \exp(-\sigma(T-t)W_t + a_2(t)) \\
&= \exp(-\sigma(T-t)W_t^* + a_2^*(t))
\end{aligned}
$$

where

$$
\begin{aligned}
a_1(t) &= \int_0^t f(0,u)du + \frac{1}{2}\sigma^2 t^3, \\
a_2(t) &= -\int_t^T f(0,u)du - \frac{\sigma^2}{2}(T-t)t(T+t), \\
a_2^*(t) &= -\int_t^T f(0,u)du - \frac{\sigma^2}{2}T(T-t)t.
\end{aligned}
$$

Denoting $P(t,T)$ by P_t, we have

$$
A = \{\frac{\mathbf{dP}}{\mathbf{dP}^*} > aH\} = \{P_t^{-\frac{t}{T-t}} > \tilde{a}(P_t - k)^+\} = \{P_t < c\}.
$$

To determine the constant c, we use the constraint condition

$$
\tilde{V}_0 = \mathbf{E}_{\mathbf{P}^*}[HI_A]
$$

where $H = B_t^{-1}(P(t,T)-k)^+$. So we have

$$\begin{aligned}\tilde{V}_0 &= \mathbf{E}_{\mathbf{P}^*}[H] - \mathbf{E}_{\mathbf{P}^*}[HI_{\{P_t>c\}}]\\
&= P(0,T)\Big\{F\Phi(\frac{\log\frac{F}{k}+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) - k\Phi(\frac{\log\frac{F}{k}-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}})\\
&\quad - F\Phi(\frac{\log\frac{F}{c}+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) + k\Phi(\frac{\log\frac{F}{c}-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}})\Big\}\end{aligned}$$

where

$$F = \frac{P(0,T)}{P(0,t)}.$$

Now $\mathbf{P}(A)$ is calculated by

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(P_t < c)\\
&= \mathbf{P}(-W_t < \frac{\log c - a_2(t)}{\sigma(T-t)})\\
&= \Phi\Big(\frac{\log c + \int_t^T f(0,u)du + \frac{\sigma^2(T-t)t(T+t)}{2}}{\sigma(T-t)\sqrt{t}}\Big).\end{aligned}$$

From the above discussions, we obtain the following proposition.

Proposition 1 (Quantile hedging for bond options)**.** *Under the Ho-Lee model, let assume that $\gamma_s = \sigma s$. Consider a quantile hedging problem for a bond call option on T-bond , struck at k with exercise time t such that a discounted option value H is $(P(t,T)-k)^+/B_t$. Then for a given $\tilde{V}_0(< H_0)$, we have the probability of the success set A*

$$\mathbf{P}(A) = \Phi\Big(\frac{\log c + \int_t^T f(0,u)du + \frac{\sigma^2(T-t)t(T+t)}{2}}{\sigma(T-t)\sqrt{t}}\Big)$$

where Φ is the distribution function of the standard normal ditribution $N(0,1)$. The constant c is given by

$$\begin{aligned}&P(0,T)\Big(F\Phi(\frac{\log\frac{F}{k}+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) - k\Phi(\frac{\log\frac{F}{k}-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}})\\
&\quad - F\Phi(\frac{\log\frac{F}{c}+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) + k\Phi(\frac{\log\frac{F}{c}-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}})\Big)\\
&= \tilde{V}_0\end{aligned}$$

with

$$F = \frac{P(0,T)}{P(0,t)} = \frac{\exp(-\int_0^T f(0,u)du)}{\exp(-\int_0^t f(0,u)du)} = \exp(-\int_t^T f(0,u)du).$$

3. Quantile hedging for caplets

In this section we deal with quantile hedging for caplets. Let $\delta > 0$. A δ-caplet is a contingent claim $\delta \times (L(T-\delta) - k)^+$ with exercise time T where $L(T-\delta)$ is the δ-period LIBOR rate set at time $T-\delta$ and k is the fixed rate, and so

$$H = B_T^{-1}\delta(L(T-\delta) - k)^+.$$

The time-0 value of the caplet is

$$H_0 = \mathbf{E}_{\mathbf{P}^*}[B_T^{-1}\delta(L(T-\delta) - k)^+].$$

Note that

$$\begin{aligned}\delta(L(T-\delta) - k)^+ &= (P(T-\delta, T)^{-1} - 1 - k\delta)^+ \\ &= (1 + k\delta)P(T-\delta, T)^{-1}((1 + k\delta)^{-1} - P(T-\delta, T))^+.\end{aligned}$$

Let $K = (1 + k\delta)^{-1}$. We have

$$\begin{aligned}H_0 &= (1 + k\delta)\mathbf{E}_{\mathbf{P}^*}[B_T^{-1}P(T-\delta, T)^{-1}(K - P(T-\delta, T))^+] \\ &= (1 + k\delta)\mathbf{E}_{\mathbf{P}^*}[\mathbf{E}_{\mathbf{P}^*}[B_T^{-1}P(T-\delta, T)^{-1}(K - P(T-\delta, T))^+|\mathcal{F}_{T-\delta}]] \\ &= (1 + k\delta)\mathbf{E}_{\mathbf{P}^*}[B_{T-\delta}^{-1}(K - P(T-\delta, T))^+] \\ &= (1 + k\delta)(K\Phi\Big(-\frac{\log\frac{F}{K} - \frac{1}{2}(\sigma\delta)^2(T-\delta)}{\sigma\delta\sqrt{T-t}}) - F\Phi(-\frac{\log\frac{F}{K} + \frac{1}{2}(\sigma\delta)^2(T-\delta)}{\sigma\delta\sqrt{T-t}})\Big)\end{aligned}$$

where

$$F = \frac{P(0, T)}{P(0, T-\delta)}.$$

Fix $\tilde{V}_0(< H_0)$. A quantile hedging of the above caplet with the initial cost V_0 is approximately calculated as follows:

The success set A is

$$\begin{aligned}A &= \{\frac{\mathbf{dP}}{\mathbf{dP}^*} > aH\} \\ &= \{\frac{\mathbf{dP}}{\mathbf{dP}^*} > aB_T^{-1}(P(T-\delta, T)^{-1} - (1 + k\delta))^+\} \\ &= \{B_T\frac{\mathbf{dP}}{\mathbf{dP}^*} > a(P(T-\delta, T)^{-1} - (1 + k\delta))^+\}.\end{aligned}$$

We will approximate the set A by $\tilde{A}$ where

$$\tilde{A} = \Big\{\mathbf{E}_{\mathbf{P}^*}\left[B_T\frac{\mathbf{dP}}{\mathbf{dP}^*}\,\Big|\,\mathcal{F}_{T-\delta}\right] > a(P(T-\delta, T)^{-1} - (1 + k\delta))^+\Big\}.$$

Let us consider again the case where

$$\gamma_s = \sigma s.$$

Then by Fact 4,

$$
\begin{aligned}
B_T \frac{\mathbf{dP}}{\mathbf{dP}^*} &= \exp\left(\sigma T W_T + a_1(T)\right) \\
&= \exp\Big(\sigma T W_T^* - \sigma T \int_0^T \gamma_s ds + a_1(T)\Big).
\end{aligned}
$$

It holds that

$$
\begin{aligned}
\mathbf{E}_{\mathbf{P}^*}\left[B_T \frac{dP}{dQ}\Big|\mathcal{F}_{T-\delta}\right] &= \exp\Big(\sigma T W_{T-\delta}^* + \frac{\sigma^2 T^2 \delta}{2} \\
&\quad - \sigma T \int_0^{T-\delta} \gamma_s ds - \frac{\sigma\delta(2T-\delta)T}{2} + a_1(T)\Big) \\
&= \exp\Big(\sigma T W_{T-\delta} + \int_0^T f(0,u)du + \frac{\sigma^2 T(T^2 - T\delta + \delta^2)}{2}\Big).
\end{aligned}
$$

By Fact 2,

$$
\begin{aligned}
P(T-\delta, T)^{-1} &= \exp\left(\sigma\delta W_{T-\delta} - a_2(T-\delta)\right) \\
&= \exp\left(\sigma\delta W_{T-\delta} + \int_{T-\delta}^T f(0,u)du + \frac{\sigma^2\delta(T-\delta)(2T-\delta)}{2}\right) \\
&= \exp\left(\sigma\delta W_{T-\delta}^* - a_2^*(T-\delta)\right)
\end{aligned}
$$

Denote $P(T-\delta,T)^{-1}$ by $P_{T-\delta}^{-1}$. Our approximating success set $\tilde{A}$ has the form

$$
\tilde{A} = \{(P_{T-\delta}^{-1})^{\frac{T}{\delta}} > \tilde{a}(P_{T-\delta}^{-1} - (1+k\delta))^+\}.
$$

The calculation goes as Föllmer and Leukert ([2] Section 3). Note that $0 < \delta < T$. We have

$$
\begin{aligned}
\tilde{A} &= \{P_{T-\delta}^{-1} < c_1\} \cup \{P_{T-\delta}^{-1} > c_2\} \\
&= \{W_{T-\delta} < b_1\} \cup \{W_{T-\delta} > b_2\}
\end{aligned}
$$

where c_1, c_2, b_1, b_2 are some constants.

From the above discussions, we obtain the following proposition.

Proposition 2 (Quantile hedging for caplets)**.** *Under the Ho-Lee model, let assume that $\gamma_s = \sigma s$. Consider a quantile hedging problem for a caplet with exercise time T such that the discounted value H is*

$$
H = B_T^{-1}\delta(L(T-\delta) - k)^+
$$

where $L(T-\delta)$ is the δ-period LIBOR rate set at time $T-\delta$.

For a given $\tilde{V}_0(< H_0)$, the approximate success set $\tilde{A}$ has the form

$$
\tilde{A} = \{P_{T-\delta}^{-1} < c_1\} \cup \{P_{T-\delta}^{-1} > c_2\} = \{W_{T-\delta} < b_1\} \cup \{W_{T-\delta} > b_2\}
$$

where $c_1 < c_2$ are two distict solutions of the equation for $P_{T-\delta}^{-1}$ such that

$$
(P_{T-\delta}^{-1})^{\frac{T}{\sigma\delta}} = \tilde{a}(P_{T-\delta}^{-1} - (1+\delta k))^+
$$

and where the constant $\tilde{a}$ is determined by the condition $\mathbf{E_Q}[HI_{\tilde{A}}] = \tilde{V}_0$. *We have*

$$\mathbf{P}(\tilde{A}) = \Phi\Big(\frac{b_1}{\sqrt{T-\delta}}\Big) + \Phi\Big(-\frac{b_2}{\sqrt{T-\delta}}\Big)$$

where Φ *is the distribution function of the standard normal ditribution* $N(0,1)$.

It means that

$$\mathbf{P}(A) \geq \mathbf{P}(\tilde{A}) = \Phi\Big(\frac{b_1}{\sqrt{T-\delta}}\Big) + \Phi\Big(-\frac{b_2}{\sqrt{T-\delta}}\Big)$$

for any optimal success set A, so $\Phi\Big(\frac{b_1}{\sqrt{T-\delta}}\Big) + \Phi\Big(-\frac{b_2}{\sqrt{T-\delta}}\Big)$ *is a lower bound of the success probability.*

4. Optimal partial hedging with the expected shortfall for bond options

In Section 2 quantile hedging for the Ho-Lee bond options is considered. In this section optimal partial hedging with the expected shortfall for the Ho-Lee bond options is investigated.

Fix $\tilde{V}_0 (< H_0)$. Minimizing the expected shortfall of the above bond call option with the initial cost $\tilde{V}_0$ is calculated as follows ([2]):

The success set A is

$$A = \{\frac{\mathbf{dP}}{\mathbf{dP^*}} > a\}$$

where

$$\frac{\mathbf{dP}}{\mathbf{dP^*}} = \exp\big(\int_0^t \gamma_s dW_s + \int_0^t \frac{\gamma_s^2 ds}{2}\big)$$

by Fact 4.

For further investigation, we consider the case when

$$\gamma_s = \gamma$$

where γ is a constant, that is the case when the market price of risk is constant.

Then

$$\int_0^t \gamma_s dW_s + \int_0^t \frac{\gamma_s^2 ds}{2} = \gamma W_t + \frac{\gamma^2 t}{2}$$

and so

$$\frac{\mathbf{dP}}{\mathbf{dP^*}} = \exp\big(\gamma W_t + \frac{\gamma^2 t}{2}\big)$$

and

$$\begin{aligned} P(t,T) &= \exp\big(-\sigma(T-t)W_t + a_2(t)\big) \\ &= \exp\big(-\sigma(T-t)W_t^* + a_2^*(t)\big) \end{aligned}$$

where

$$a_2(t) = -\int_t^T f(0,u)du - \sigma(T-t)t(\frac{\sigma T}{2} + \gamma),$$
$$a_2^*(t) = -\int_t^T f(0,u)du - \frac{\sigma^2}{2}T(T-t)t.$$

Denoting $P(t,T)$ by P_t, we have

$$A = \{\frac{\mathbf{dP}}{\mathbf{dP}^*} > a\} = \{P_t^{-\frac{\gamma}{\sigma(T-t)}} > \tilde{a}\} = \{P_t < c\}.$$

To determine the constant c, we use the constraint condition

$$\tilde{V}_0 = \mathbf{E}_{\mathbf{P}^*}[HI_A]$$

where $H = B_t^{-1}(P(t,T)-k)^+$. It means that

$$\begin{aligned}\tilde{V}_0 &= \mathbf{E}_{\mathbf{P}^*}[H] - \mathbf{E}_{\mathbf{P}^*}[HI_{\{P_t>c\}}] \\ &= P(0,T)\Big\{F\Phi(\frac{\log F - \log k + \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}) - k\Phi(\frac{\log F - \log k - \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}) \\ &\quad - F\Phi(\frac{\log F - \log c + \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}) + k\Phi(\frac{\log F - \log c - \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}})\Big\}\end{aligned}$$

where

$$F = \frac{P(0,T)}{P(0,t)}.$$

Now the minimal expected shortfall $L(\tilde{V}_0)$ is calculated by

$$\begin{aligned}L(\tilde{V}_0) &= \mathbf{E}_{\mathbf{P}}[B_t^{-1}(P(t,T)-k)^+ I_{P_t>c}] \\ &= B(G\Phi\Big(\frac{G - \log c + \frac{1}{2}\sigma^2(T-t)t}{\sigma\sqrt{(T-t)t}}\Big) - k\Phi\Big(\frac{G - \log c - \frac{1}{2}\sigma^2(T-t)t}{\sigma\sqrt{(T-t)t}}\Big))\end{aligned}$$

where

$$G = \frac{1}{2}\sigma^2(T-t)t^2 - \int_t^T f(0,u)du - \sigma\gamma(T-t)t$$

and

$$B = \exp(-(\int_0^t f(0,u)du + \frac{1}{2}\sigma\gamma t^2)).$$

From the above discussions, we obtain the following proposition.

Proposition 3 (Minimizing the shortfall risk hedging for bond options)**.** *Under the Ho-Lee model, let assume that* $\gamma_s = \gamma$ *where* γ *is a constant. Consider a minimizing the expected shortfall for a bond call option on T-bond , struck at k with exercise time t such that a discounted option value H is* $(P(t,T)-k)^+/B_t$*.*

Then for a given $\tilde{V}_0(< H_0)$, the minimal expected shortfall

$$\begin{aligned} L(\tilde{V}_0) &= \mathbf{E_P}[B_t^{-1}(P(t,T)-k)^+ I_{P_t>c}] \\ &= B(G\Phi\Big(\frac{G-\log c+\frac{1}{2}\sigma^2(T-t)t}{\sigma\sqrt{(T-t)t}}\Big) - k\Phi\Big(\frac{G-\log c-\frac{1}{2}\sigma^2(T-t)t}{\sigma\sqrt{(T-t)t}}\Big)) \end{aligned}$$

where

$$G = \frac{1}{2}\sigma^2(T-t)t^2 - \int_t^T f(0,u)du - \sigma\gamma(T-t)t$$

and

$$B = \exp(-(\int_0^t f(0,u)du + \frac{1}{2}\sigma\gamma t^2)).$$

The constant c is given by

$$\begin{aligned} P(0,T)\Big\{&F\Phi(\frac{\log F-\log k+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) - k\Phi(\frac{\log F-\log k-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) \\ &- F\Phi(\frac{\log F-\log c+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}) + k\Phi(\frac{\log F-\log c-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}})\Big\} \\ &= \tilde{V}_0 \end{aligned}$$

with

$$F = \frac{P(0,T)}{P(0,t)} = \exp(-\int_t^T f(0,u)du).$$

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債権オプションに対するクォンタイル・ヘッジおよび期待不足額の最小化

辻　井　芳　樹

要　旨

1999 年および 2000 年の論文で Föllmer と Leukert は，株オプションに関するクォンタイル・ヘッジおよび期待不足額の最小化について研究している．

本論文の前半では，債権オプションとキャプレットに対するクォンタイル・ヘッジについて調べる．リスク市場価格 γ_t がボラティリティ σ で $\sigma t + c$（c は定数）と表される場合の Ho-Lee 債権モデルに対して，成功確率の式を与える．またキャプレットについても同様な仮定の下で成功確率の下界を与える．

論文の後半では債権オプションに対する期待不足額の最小化について調べる．リスク市場価格 γ_t が定数である場合の Ho-Lee 債権モデルに対して，最小期待不足額の式を与える．

キーワード： クォンタイル・ヘッジ，期待不足額の最小化，債権オプション，キャプレット，Ho-Lee債権モデル