Lonesum Matrices and Poly-Bernoulli Numbers

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Abstract

A lonesum matrix is a matrix with entries 0 or 1 which is uniquely determined by its column sum and row sum. In this paper, we show that the number $L(m, n)$ of m-by-n lonesum matrices and the poly-Bernoulli number $\mathbb{B}_n^{(-m)}$ satisfy the same recurrence relations. Using these relations, we give a new proof of Brewbaker's formula $L(m, n) = \mathbb{B}_n^{(-m)}$.

Keywords: (0, 1)-matrix,lonesum matrix, poly-Bernoulli number, Brewbaker's formula, recurrence relations

1. Introduction

A lonesum matrix is a matrix with entries 0 or 1 which is uniquely determined by its column sum and row sum. It is an interesting problem in combinatorial matrix theory to study the number $L(m, n)$ of m-by-n lonesum matrices. Brewbaker [1] showed that $L(m, n)$ coincides with the poly-Bernoulli number $\mathbb{B}_n^{(-m)}$ introduced by Kaneko [3] as a generalization of the Bernoulli number. The main object of this paper is to prove that $L(m, n)$ and $\mathbb{B}_n^{(-m)}$ satisfy the same recurrence relations. As a by-product, we give a new proof of Brewbaker's formula.

This paper is organized as follows. Section 2 is of preliminary nature. We recall the definitions of the binomial number and the Stirling subset number. We also recall several fundamental properties of (0, 1)-matrices. In Section 3, after recalling the definition and basic properties of lonesum matrices, we give recurrence relations for $L(m, n)$. In Section 4, we recall the definition of poly-Bernoulli numbers and prove recurrence relations for $\mathbb{B}_n^{(-m)}$. A new proof of Brewbaker's formula is given in Section 5 by combining the results of Sections 3 and 4.

Notation

For a finite set X, we denote by $\#(X)$ the cardinality of X. We deneto by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers.

2. Preliminaries

2.1 Binomial numbers and Stirling subset numbers

For $m, n \in \mathbb{Z}$, we define the binomial number $\binom{n}{m}$ by

$$\binom{n}{m} := \begin{cases} \dfrac{n!}{m!\,(n-m)!} & \text{if } n \geq m \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we put

$$n! := \begin{cases} 1 \times 2 \times \cdots \times n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

We easily see that

$$\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m} \tag{1}$$

holds for $m, n \in \mathbb{Z}_{\geq 0}$.

For $m, n \in \mathbb{Z}$, we define the Stirling subset number $\left\{ {n \atop m} \right\}$ (or the Stirling number of the second kind) by

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop m} \right\} = 0 \quad (n, m \neq 0),$$

and

$$\left\{ {n+1 \atop m} \right\} = \left\{ {n \atop m-1} \right\} + m \left\{ {n \atop m} \right\}.$$

When $m, n > 0$, $\left\{ {n \atop m} \right\}$ is equal to the number of ways of partitioning a set of n elements into m nonempty sets. The following fact is well-known.

Proposition 2.1. *For $m, n \in \mathbb{Z}_{\geq 0}$, we have*

$$\left\{ {n \atop m} \right\} = \frac{(-1)^m}{m!} \sum_{l=0}^{m} (-1)^l \binom{m}{l} l^n.$$

Here we make a convention that $l^0 = 1$ for $l \in \mathbb{Z}_{\geq 0}$.

2.2 Partitions

Let N be a positive integer and $(p_1, \ldots, p_m) \in (\mathbb{Z}_{\geq 0})^m$. If $p_1 \geq \ldots \geq p_m$ and $p_1 + \ldots + p_m = N$, we call $(p_1, \ldots, p_m)$ a partition of N. For $s = (s_1, \ldots, s_m) \in (\mathbb{Z}_{\geq 0})^m$ with $s_1 + \cdots + s_m = N$, there uniquely exists a partition $p = (p_1, \ldots, p_m)$ of N such that $p_i = s_{\sigma(i)}$ with some permutation σ of $\{1, \ldots, m\}$. In this situation, we say that p is the partition of N associated with s.

2.3 $(0, 1)$-Matrices

A matrix A is called a $(0, 1)$-matrix if each entry of A is either zero or one. Let $\mathcal{B}(m, n)$ be the set of m-by-n $(0,1)$-matrices. Then $\#\mathcal{B}(m,n) = 2^{mn}$. The sum of all entries of $A = (a_{ij}) \in \mathcal{B}(m,n)$ is denoted by $N(A)$. Namely

$$N(A) := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$

For $A \in \mathcal{B}(m,n)$, we put

$$c(A) := (c_1(A), \dots, c_m(A)),\ \ c_i(A) := \sum_{j=1}^{n} a_{ij},$$

$$r(A) := (r_1(A), \dots, r_n(A)),\ \ r_j(A) := \sum_{i=1}^{m} a_{ij}.$$

We call $c(A)$ (respectively $r(A)$) the column (respectively row) sum of A. Note that

$$\sum_{i=1}^{m} c_i(A) = \sum_{j=1}^{n} r_j(A) = N(A).$$

Example 2.2. If $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, we have $c(A) = (2,1)$ and $r(A) = (1,2,0)$.

A matrix $A = (a_{ij}) \in \mathcal{B}(m,n)$ is called a Ferrars matrix if the conditions

(i) $a_{ij} = 0 \Rightarrow a_{kj} = 0 \ \ (k \geq i)$,

(ii) $a_{ij} = 0 \Rightarrow a_{il} = 0 \ \ (l \geq j)$

are satisfied. Denote by $\mathcal{F}(m,n)$ the set of m-by-n Ferrars matrices.

Example 2.3.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a Ferrars matrix.

The following fact is well-known and easily verified.

Proposition 2.4. (1) *If $A \in \mathcal{F}(m,n)$, $c(A)$ and $r(A)$ are partitions of $N(A)$.*

(2) *We have*

$$\#\mathcal{F}(m,n) = \binom{m+n}{m}.$$

Let $p = (p_1, \dots, p_m)$ be a partition of $N \in \mathbb{Z}_{>0}$. For a positive integer n, put

$$\pi_n(p) := (p_1^*, \dots, p_n^*),$$

where

$$p_i^* = \#\{j \mid 1 \leq j \leq m, p_j \geq i\}.$$

Then $\pi_n(p)$ is also a partition of N if $n \geq p_1$. If $n \geq p_1$, there uniquely exists an m-by-n Ferrars matrix A such that $c(A) = p$. For such an A, we have $r(A) = \pi_n(p)$.

The following fundamental result was shown by Gale [2] and Ryser [6] (see also [5]).

Theorem 2.5. *Let $s = (s_1, \dots, s_m) \in (\mathbb{Z}_{\geq 0})^m$ and $t = (t_1, \dots, t_n) \in (\mathbb{Z}_{\geq 0})^n$ satisfying $s_1 + \cdots + s_m = t_1 + \cdots + t_n = N$. Let $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_n)$ be the partitions of N associated with s and t respectively, and let $\pi_n(p) = (p_1^*, \dots, p_n^*)$. Then there exists an $A \in \mathcal{B}(m, n)$ such that $c(A) = s$ and $r(A) = t$ if and only if the following conditions are satisfied:*

$$\begin{aligned} p_1^* &\geq q_1, \\ p_1^* + p_2^* &\geq q_1 + q_2, \\ &\vdots \\ p_1^* + \cdots + p_n^* &\geq q_1 + \cdots + q_n. \end{aligned}$$

Example 2.6. Let $p = (3, 1)$ and $q = (2, 2, 0)$. Then $\pi_3(p) = (2, 1, 1)$ and the condition of Theorem 2.5 is not satisfied. Thus there is no $A \in \mathcal{B}(2, 3)$ such that $c(A) = p$ and $r(A) = q$.

3. Lonesum matrices

3.1 The definition of lonesum matrices

An m-by-n $(0, 1)$-matrix A is called a lonesum matrix if the following condition holds:

$$A' \in \mathcal{B}(m, n),\ c(A') = c(A),\ r(A') = r(A) \implies A' = A.$$

Namely a lonesum matrix is a $(0, 1)$-matrix which is uniquely determined by its column sum and row sum. We denote by $\mathcal{L}(m, n)$ the set of m-by-n lonesum matrices.

Example 3.1. We have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{L}(2, 3).$$

On the other hand, since

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

have the same column sum $(1, 1)$ and row sum $(1, 1, 0)$, we have $B, C \notin \mathcal{L}(m, n)$.

3.2 Criterions

The following result is due to Ryser [6].

Theorem 3.2. *For $A \in \mathcal{B}(m, n)$, the following conditions are equivalent.*

(i) *A is a lonesum matrix.*

(ii) *A has no minor of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

(iii) *A is obtained from a Ferrars matrix by permutations of columns and rows.*

(iv) *Let p and q be the partitions of $N(A)$ associated with $c(A)$ and $r(A)$, respectively. Then $\pi_n(p) = q$.*

Corollary 3.3. *Let A be a lonesum matrix.*

(1) *A matrix obtained by permutations of columns and rows of A is a lonesum matrix.*

(2) *The transpose of A is a lonesum matrix.*
(3) *Any minor of A is a lonesum matrix.*

3.3 Recurrence relations

For $m, n \in \mathbb{Z}_{>0}$, we put $L(m,n) = \#\mathcal{L}(m,n)$. By Corollary 3.3 (2), we have $L(m,n) = L(n,m)$. We make a convention that $L(m,0) = L(0,n) = L(0,0) = 1 \ \ (m,n > 0)$. We now state one of the main results of this paper.

Theorem 3.4. *For $m, n \in \mathbb{Z}_{>0}$, we have*

$$L(m,n) = \sum_{r=0}^{m-1} (-1)^{m+r-1} \binom{m}{r} \sum_{k=0}^{n} \binom{n}{k} L(r,k).$$

To prove Theorem 3.4, we need several preparations. For $0 \leq k \leq n$, set

$$\mathcal{M}(m,n,k) := \{A \in \mathcal{L}(m,n) \mid \min_{1 \leq i \leq m} c_i(A) = n-k\},$$
$$M(m,n,k) := \#\mathcal{M}(m,n,k).$$

Lemma 3.5. *We have*

$$L(m,n) = \sum_{k=0}^{n} M(m,n,k).$$

Proof. This follows from

$$\mathcal{L}(m,n) = \bigcup_{k=0}^{n} \mathcal{M}(m,n,k) \qquad \text{(disjoint union)}.$$

□

Theorem 3.4 is a direct consequence of Lemma 3.5 and the following.

Proposition 3.6. *We have*

$$M(m,n,k) = \binom{n}{k} \sum_{r=0}^{m-1} (-1)^{m+r-1} \binom{m}{r} L(r,k).$$

To prove the proposition, we fix such a triple (m,n,k) and put

$$\mathcal{A}_i := \{A \in \mathcal{M}(m,n,k) \mid c_i(A) = n-k\} \qquad (i = 1, \ldots, m).$$

Note that

$$\mathcal{A}_i = \{A \in \mathcal{L}(m,n) \mid c_i(A) = n-k,\ c_j(A) \geq n-k\ (1 \leq j \leq m)\}.$$

Lemma 3.7.

$$M(m,n,k) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \#(\mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}).$$

Proof. This follows from $\mathcal{M}(m,n,k) = \bigcup_{i=1}^{m} \mathcal{A}_i$ and the inclusion-exclusion argument. □

Proposition 3.8. *we have*

$$\#(\mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}) = \binom{n}{k} L(m-r,k), \ (1 \leq i_1 < \cdots < i_r \leq m). \tag{2}$$

We postpone the proof of Proposition 3.8 until the next subsection. The equality (2) implies

$$\sum_{1 \leq i_1 < \cdots < i_r \leq m} \#(\mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}) = \binom{m}{r}\binom{n}{k} L(m-r,k). \tag{3}$$

Proposition 3.8 now follows from (3) and Lemma 3.7 .

3.4 Proof of Proposition 3.8

Let $I = \{i_1, \ldots, i_r\}$ and put $I^c = \{1, \ldots, m\} \setminus I = \{i'_1, \ldots, i'_{m-r}\}$ $(1 \leq i'_1 < \cdots < i'_{m-r} \leq m)$. We denote by $\mathcal{J}$ the set of $(n-k)-$tuples of integers $\{j_1, \ldots, j_{n-k}\}$ with $1 \leq j_1 < \cdots < j_{n-k} \leq n$. We put $J^c = \{1, \ldots, n\} \setminus J$ for $J \in \mathcal{J}$.

We first suppose that $k = n$. In this case, $\mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}$ is the set of $A = (a_{ij}) \in \mathcal{L}(m,n)$ with $a_{i_1 j} = \cdots = a_{i_r j} = 0$ $(j = 1, \ldots, n)$. It is easily verifisd that $\#(\mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}) = L(m-r,n)$, which proves (3.1).

We henceforth suppose that $k < n$. Suppose that $r = m$. For $J \in \mathcal{J}$, let $A^J = (a^J_{ij}) \in \mathcal{B}(m,n)$, where

$$a^J_{ij} = \begin{cases} 1 & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_m} &= \mathcal{A}_1 \cap \cdots \cap \mathcal{A}_m \\ &= \{A \in \mathcal{L}(m,n) | c_i(A) = n-k \ (1 \leq i \leq m)\} \\ &= \{A^J | J \in \mathcal{J}\}. \end{aligned}$$

Hence

$$\#(\mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_m}) = \#\mathcal{J} = \binom{n}{n-k} = \binom{n}{k} = \binom{n}{k} L(0,k),$$

which proves (3.1).

Lemma 3.9. *Suppose that* $1 \leq r < m$.

(1) *For* $A = (a_{ij}) \in \mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}$, *there exists a unique element* J *of* $\mathcal{J}$ *such that*

$$\begin{cases} a_{i,j} &= 1 \quad (1 \leq i \leq m, j \in J), \\ a_{i_\alpha,j} &= 0 \quad (1 \leq \alpha \leq r, j \in J^c). \end{cases}$$

(2) *Let* A *and* J *as in* (1). *Then* $A' := (a_{ij})_{i \notin I, j \notin J}$ *belongs to* $\mathcal{L}(m-r,k)$.

(3) *The mapping*

$$\varphi\colon \mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r} \ni A \mapsto A' \in \mathcal{L}(m-r,k)$$

defined as in (2) *is surjective and*

$$\#\left(\varphi^{-1}(A')\right) = \binom{n}{k}$$

for every $A' \in \mathcal{L}(m-r, k)$.

Proof. Let $A = (a_{ij}) \in \mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}$. Then there exists an element J of $\mathcal{J}$ such that

$$\begin{aligned} a_{i_1,j} &= 1 \qquad (j \in J), \\ a_{i_1,j} &= 0 \qquad (j \in J^c). \end{aligned}$$

Suppose that there exist $i \quad (1 \le i \le m)$ and $\beta \quad (1 \le \beta \le n-k)$ with $a_{i,j_\beta} = 0$. Since $c_i(A) \ge n-k$, there exists $j \in J^c$ such that $a_{i,j} = 1$. Then we have

$$\begin{pmatrix} a_{i_1,j_\beta} & a_{i_1,j} \\ a_{i,j_\beta} & a_{i,j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This contradicts the assumption that A is a lonesum matrix (cf. Theorem 3.2(2)) and hence we have proved $a_{i,j} = 1$ $(1 \le i \le m, j \in J)$.

Next suppose that there exist α, j $(2 \le \alpha \le r, j \in J^c)$ with $a_{i_\alpha,j} = 1$. Since $c_{i_\alpha}(A) = n-k$, there exists β $(1 \le \beta \le n-k)$ such that $a_{i_\alpha,j_\beta} = 0$. Then

$$\begin{pmatrix} a_{i_1,j_\beta} & a_{i_1,j} \\ a_{i_\alpha,j_\beta} & a_{i_\alpha,j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

a contradiction. Thus we have proved $a_{i_\alpha,j} = 0$ $(1 \le \alpha \le r, j \in J^c)$, which completes the proof of the first assertion of the lemma.

The second assertion follows from Corollary 3.3 (3).

For $A' = (a'_{\alpha\beta})_{1\le\alpha\le m-r, 1\le\beta\le k} \in \mathcal{L}(m-r, k)$ and $J \in \mathcal{J}$, define an m-by-n $(0, 1)$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } j \in J, \\ 0 & \text{if } i \in I \text{ and } j \in J^c, \\ a'_{\alpha,\beta} & \text{if } i = i'_\alpha \text{ and } j = j'_\beta. \end{cases}$$

Then we have $A \in \mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_r}$. We write $\psi(A', J)$ for A. Then $\varphi(\psi(A', J)) = A'$, which shows the surjectivity of φ. By (1), we have $\varphi^{-1}(A') = \{\psi(A', J) | J \in \mathcal{J}\}$, and hence $\#(\varphi^{-1}(A')) = \#\mathcal{J} = \binom{n}{k}$, which proves (3).

□

Proposition 3.8 in the case $k < n$ and $r < m$ is a direct consequene of Lemma 3.9 (3).

4. Poly-Bernoulli numbers

4.1 The definition of poly-Bernoulli numbers

The poly-Bernoulli numbers were introduced by Kaneko ([3]; see also [4]). For every integer k, we define the poly-Bernoulli number $\mathbb{B}_n^{(k)} \in \mathbb{Q}$ $(n = 0, 1, \ldots)$ by

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)} \frac{t^n}{n!}.$$

Here $Li_k(t)$ denotes a formal power series $\displaystyle\sum_{n=1}^{\infty} \frac{t^n}{n^k}$.

Theorem 4.1 ([3]).

(1) *For $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, we have*

$$\mathbb{B}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k}.$$

In particular, $\mathbb{B}_n^{(-k)}$ is a positive integer if $k \geq 0$.

(2) *For $n, k \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}.$$

4.2 Recurrence relations for poly-Bernoulli numbers with negative upper index

We now state the second main result of this paper.

Theorem 4.2. *We have*

$$\mathbb{B}_n^{(-m)} = \sum_{r=0}^{m-1} (-1)^{m+r-1} \binom{m}{r} \sum_{k=0}^{n} \binom{n}{k} \mathbb{B}_k^{(-r)} \qquad (n, m \in \mathbb{Z}_{>0}). \tag{4}$$

To prove Theorem 4.2, we need the following.

Lemma 4.3. *For $m \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}$ $(0 \leq j < m)$, we have*

$$\sum_{r=j}^{m-1} \binom{m}{r} \left\{ {r \atop j} \right\} = (j+1) \left\{ {m \atop j+1} \right\}. \tag{5}$$

Proof. Since $\left\{ {r \atop j} \right\} = 0$ if $r < j$, we have

$$\sum_{r=j}^{m-1} \binom{m}{r} \left\{ {r \atop j} \right\} = \sum_{r=0}^{m-1} \binom{m}{r} \left\{ {r \atop j} \right\}.$$

By Proposition 2.1 and (1), the left-hand side of (5) is equal to

$$\begin{aligned}
&\sum_{r=0}^{m-1} \binom{m}{r} \frac{(-1)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} l^r \\
&= \frac{(-1)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \sum_{r=0}^{m-1} \binom{m}{r} l^r \\
&= \frac{(-1)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} ((l+1)^m - l^m) \\
&= \frac{(-1)^j}{j!} \left\{ \sum_{l=1}^{j+1} (-1)^{l+1} \binom{j}{l-1} l^m + \sum_{l=0}^{j} (-1)^{l+1} \binom{j}{l} l^m \right\}
\end{aligned}$$

$$= \frac{(-1)^j}{j!}\sum_{l=0}^{j+1}(-1)^{l+1}\left(\binom{j}{l-1}+\binom{j}{l}\right)l^m$$
$$= \frac{(-1)^j}{j!}\sum_{l=0}^{j+1}(-1)^{l+1}\binom{j+1}{l}l^m$$
$$= (j+1)\left\{ {m \atop j+1} \right\},$$

which completes the proof of the lemma. □

4.3 Proof of Theorem 4.2

First consider the case where $m = 1$. Then the right-hand side of (4) is equal to

$$\sum_{k=0}^{n}\binom{n}{k}\mathbb{B}_k^{(0)} = \sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

On the other hand, by Theorem 4.1, we have

$$\mathbb{B}_n^{(-1)} = \mathbb{B}_1^{(-n)} = -\left\{ {1 \atop 0} \right\} + \left\{ {1 \atop 1} \right\}2^n = 2^n,$$

which proves (4) in this case.

Next suppose that $m \geq 2$. By Theorem 4.1, the right-hand side of (4) is equal to

$$\sum_{r=0}^{m-1}(-1)^{m+r-1}\binom{m}{r}\sum_{k=0}^{n}\binom{n}{k}\mathbb{B}_r^{(-k)}$$
$$= \sum_{r=0}^{m-1}(-1)^{m+r-1}\binom{m}{r}\sum_{k=0}^{n}\binom{n}{k}\left\{(-1)^r\sum_{j=0}^{r}(-1)^j j!\left\{ {r \atop j} \right\}(j+1)^k\right\}$$
$$= (-1)^{m-1}\sum_{r=0}^{m-1}\binom{m}{r}\sum_{j=0}^{r}(-1)^j j!\left\{ {r \atop j} \right\}\sum_{k=0}^{n}\binom{n}{k}(j+1)^k$$
$$= (-1)^{m-1}\sum_{r=0}^{m-1}\binom{m}{r}\sum_{j=0}^{r}(-1)^j j!\left\{ {r \atop j} \right\}(j+2)^n$$
$$= \sum_{j=0}^{m-1}A_j,$$

where we put

$$A_j = (-1)^{m+j-1}\sum_{r=j}^{m-1}\binom{m}{r}j!\left\{ {r \atop j} \right\}(j+2)^n.$$

On the other hand, the left-hand side of (4) is equal to

$$\mathbb{B}_m^{(-n)} = \sum_{j=0}^{m}(-1)^{j+m}j!\left\{ {m \atop j} \right\}(j+1)^n = \sum_{j=0}^{m-1}B_j,$$

where we put

$$B_j = (-1)^{j+m+1}(j+1)!\left\{ {m \atop j+1} \right\}(j+2)^n.$$

By (5), we have $A_j = B_j$ $(j = 0, \ldots, m-1)$, which implies (4) in the case where $m \geq 2$. Thus the proof of Theorem 4.2 has been completed.

5. A formula of Brewbaker

5.1 Brewbaker's theorem

Brewbaker showed the following remarkable result in [1].

Theorem 5.1. *For $m, n \in \mathbb{Z}_{>0}$, we have*

$$L(m,n) = \mathbb{B}_n^{(-m)}. \tag{6}$$

In [1], Brewbaker gave three different proofs. We will give a new proof of Theorem 5.1 based on Theorems 3.4 and 4.2.

5.2 Proof of Theorem 5.1

We use the induction on m.

Since $L(1,n) = 2^n$ and $\mathbb{B}_n^{(-1)} = 2^n$, (6) holds for $m = 1$. Suppose that (6) holds for $m-1$. By Theorem 3.4, we have

$$L(m,n) = \sum_{r=0}^{m-1}(-1)^{m+r-1}\binom{m}{r}\sum_{k=0}^{n}\binom{n}{k}L(r,k).$$

By induction assumption and Theorem 4.2, we have

$$\begin{aligned} L(m,n) &= \sum_{r=0}^{m-1}(-1)^{m+r-1}\binom{m}{r}\sum_{k=0}^{n}\binom{n}{k}\mathbb{B}_k^{(-r)} \\ &= \mathbb{B}_n^{(-m)}. \end{aligned}$$

Thus Theorem 5.1 has been established.

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ロンサム行列と多重ベルヌイ数

四　方　未　来

要　旨

ロンサム行列とは，成分が0と1のみの行列で,それぞれの行を足し合わせ並べてできる列ベクトルとそれぞれの列を足し合わせ並べてできる行ベクトルによって一意的に定まる行列のことである．本論文では，m行n列のロンサム行列の総数$L(m,n)$と金子によってベルヌイ数の一般化として導入された多重ベルヌイ数$\mathbb{B}_n^{(-m)}$が同一の漸化式を満たすことを示す．これより，Brewbakerの公式$L(m,n)=\mathbb{B}_n^{(-m)}$の新しい証明を得る．

キーワード：(0, 1)-行列，ロンサム行列，多重ベルヌイ数，Brewbakerの公式，漸化式