

On p -adic analytic families of eigenforms of infinite slope in the p -supersingular case

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Abstract

Let p be an odd prime number and f a newform of weight k of infinite $T(p)$ -slope, i.e., $f|T(p) = 0$. We assume that the conductor N of f is prime to p (we denote such a case as *the p -supersingular case*). Let f^* be a p -stabilized newform of level Np originating from f which has $T(p)$ -slope $\frac{k-1}{2}$. In this study, we construct a p -adic analytic family of eigenforms with infinite $T(p)$ -slope passing through f by twisting the Coleman family of $T(p)$ -slope $\frac{k-1}{2}$ passing through f^* using the trivial Dirichlet character modulo p .

Keywords: modular form, Hecke algebra, Galois representation, p -adic family, infinite $T(p)$ -slope

1. Introduction

Let l be any prime number. We denote by $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_l$ an algebraic closure of the rational number field \mathbb{Q} and the l -adic number field \mathbb{Q}_l , respectively. Let \mathbb{C} be the complex number field. We take the l -adic completion \mathbb{C}_l of $\bar{\mathbb{Q}}_l$. Then we fix two embeddings of fields $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_l : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$, and an isomorphism $\mathbb{C}_l \xrightarrow{\sim} \mathbb{C}$ which commutes with i_∞ and i_l . Let p be an odd prime number which we fix in this article. Let ord_p be the normalized p -adic valuation on \mathbb{C}_p so that $\text{ord}_p(p) = 1$ and $|\cdot|$ the absolute value given by ord_p . Then we denote by $\mathcal{O}_{\mathbb{C}_p}$ the subring of \mathbb{C}_p consisting of elements s such that $|s| \leq 1$. We denote by \mathbb{Z} and \mathbb{R} the ring of rational integers and the field of real numbers, respectively.

Let N be a positive integer which is prime to p , $k \geq 2$ an integer and f a cuspidal normalized eigenform of level Np^ν and weight k with some integer $\nu \geq 0$.

Definition 1.1. For the usual Hecke operator $T(p)$ at p , the $T(p)$ -slope α of f is defined as

$$\alpha = \text{ord}_p(i_p(a_p(f))),$$

where $a_p(f)$ is the eigenvalue of f for $T(p)$.

Then the $T(p)$ -slope α is a non-negative rational number in the case where $a_p(f)$ is not 0. On the other hand, when $a_p(f) = 0$, we say that f has *infinite $T(p)$ -slope*.

Let $G_{\mathbb{Q}}$ be the absolute Galois group of \mathbb{Q} . It is known that there exists an irreducible continuous Galois representation

$$\rho_f : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}_f)$$

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associated to f defined over the ring \mathcal{O}_f of integers in the finite extension K_f generated by all Hecke eigenvalues of f over \mathbb{Q}_p . In particular, we have the following properties:

- (1) ρ_f is unramified outside the prime divisors of Np and ∞ .
- (2) $\text{Trace}(\rho_f(\text{Frob}_l))$ is equal to the $T(l)$ -eigenvalue of f for any prime number $l \nmid Np$, where Frob_l and $T(l)$ are the geometric Frobenius element and the usual Hecke operator at l , respectively.

Definition 1.2. We put $S_{Np} := \{\text{the prime divisors of } Np\} \cup \{\infty\}$. A family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of eigenforms $f_{k'}$ of weight k' having fixed $T(p)$ -slope α parametrized by some arithmetic progression \mathcal{K} starting from k is said to be a *p -adic analytic family of $T(p)$ -slope α passing through f* if we have $f_k = f$ at $k' = k$ and there exists an irreducible Galois representation

$$\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{I})$$

unramified outside a finite set of rational places including S_{Np} such that the family $\{\rho_{f_{k'}}\}_{k' \in \mathcal{K}}$ of Galois representations associated to the family $\{f_{k'}\}_{k' \in \mathcal{K}}$ is interpolated p -adically by ρ , i.e., for each $k' \in \mathcal{K}$, we have

$$\rho \pmod{P_{k'}} \cong \rho_{f_{k'}}.$$

Here \mathbb{I} is a finite integral extension over the affinoid algebra $A(B)$ associated to some 1-dimensional affinoid disk B defined over \mathbb{C}_p with $\mathcal{K} \subset B(\mathbb{Z})$, and $P_{k'}$ is a maximal ideal of \mathbb{I} lying over the maximal ideal in $A(B)$ corresponding to the closed point k' in B .

Remark 1.1. Let \mathfrak{m}_f and \mathbf{k}_f be the maximal ideal and the residue field of \mathcal{O}_f , respectively. We put the residual representation

$$\bar{\rho} := \rho_f \pmod{\mathfrak{m}_f} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbf{k}_f).$$

If $\bar{\rho}$ is absolutely irreducible and ρ has values in $\text{GL}_2(\mathcal{O}_K\langle\zeta\rangle)$ with the convergent power series ring $\mathcal{O}_K\langle\zeta\rangle$ over the ring \mathcal{O}_K of integers in some finite extension K over \mathbb{Q}_p with one variable ζ , then $\bar{\rho}$ is equivalent to the residual representation $\bar{\rho}_{f_{k'}}$ associated to $f_{k'}$ for any $k' \in \mathcal{K}$ because we have $\text{Trace}(\bar{\rho}) = \text{Trace}(\bar{\rho}_{f_{k'}})$ by Chebotarev density theorem. Therefore in this case, the p -adic analytic family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of $T(p)$ -slope α passing through f gives us a p -adic analytic family of modular points, a so-called *modular arc*, in the universal deformation space associated to $\bar{\rho}$ (cf. [17, Sections 17, 18]).

When $\alpha = 0$, i.e., f is *ordinary* at p , Hida [11] and [12] constructed p -adic analytic families of $T(p)$ -slope 0 passing through given ordinary eigenforms by investigating p -adic ordinary Hecke algebras of level p^{∞} . The result of Hida has been generalized to the case of any finite $T(p)$ -slope α by Coleman [5], [6] and [7]. He investigated the p -adic Riesz theory for a certain completely continuous operator U acting on families of p -adic Banach spaces consisting of p -adic modular forms to construct p -adic analytic families of eigenforms of $T(p)$ -slope α passing through given eigenforms. Since the Newton polygon of the characteristic power series of the specialized operator U_k of weight k played an important role, the finiteness condition of $T(p)$ -slopes was very essential in his construction.

Remark 1.2. On the page 467 of [6], only an outline of a proof of [6, Corollary B5.7.1] insisting the existence of Coleman families of any finite $T(p)$ -slope is given. Since the statement of his families has been reprised and used by many people, we would like to write down a detailed proof in Section 1.2 for confirmation (cf. Corollary 2.3 (1) below).

In this context, it is very natural to ask if it is possible to construct a p -adic analytic family of eigenforms of infinite $T(p)$ -slope passing through a given eigenform or not. In this article, we shall construct such a family passing through a given newform of infinite $T(p)$ -slope whose conductor is prime to p by twisting the Coleman family of $T(p)$ -slope $\frac{k-1}{2}$ passing through a p -stabilized newform coming from f by the trivial Dirichlet character $\mathbf{1}_p$ modulo p as in the following

Theorem 1.1. *Let p be an odd prime number, N a positive integer which is prime to p and $k \geq 2$ an integer. We denote by $\mathbf{1}_p$ the trivial Dirichlet character modulo p . Let f be a cuspidal normalized newform of conductor N , weight k and character ε with Fourier expansion $f(q) = \sum_{n \geq 1} a_n(f)q^n$. We assume that f has infinite $T(p)$ -slope. Then there exists a formal power series*

$$F(q) := \sum_{n \geq 1, (n,p)=1} a_n q^n$$

in the indeterminate q with a family $\{a_n\}_{n \geq 1, (n,p)=1} \subset A(B)$ of analytic functions on the 1-dimensional affinoid disk B of radius p^{-m} over \mathbb{C}_p around the center k with some integer $m \geq 0$ such that for the arithmetic progression \mathcal{K} of radius $(p-1)p^m$ starting from k , the specialized formal power series

$$F_{k'}(q) := \sum_{n \geq 1, (n,p)=1} a_n(k')q^n$$

at each $k' \in \mathcal{K}$ gives the Fourier expansion of a cuspidal normalized eigenform $F_{k'}$ of weight k' , level Np^2 and character ε with infinite $T(p)$ -slope. Especially, the specialized eigenform F_k at the initial weight k is the twisted eigenform $f \otimes \mathbf{1}_p$ as

$$F_k(q) = \sum_{n \geq 1, (n,p)=1} a_n(f)q^n.$$

Moreover, the family $\{\rho_{F_{k'}}\}_{k' \in \mathcal{K}}$ of Galois representations associated to $F_{k'}$'s is interpolated p -adically by an irreducible Galois representation

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(A(B))$$

unramified outside the finite set S_{Np} and satisfying for each prime number $l \notin S_{Np}$,

$$\mathrm{Trace}(\rho(\mathrm{Frob}_l)) = a_l,$$

i.e., we have for any $k' \in \mathcal{K}$,

$$\rho \pmod{P_{k'}} \cong \rho_{F_{k'}}.$$

Therefore putting $f_k := f$ and $f_{k'} := F_{k'}$ for any $k' \in \mathcal{K} \setminus \{k\}$, the family $\{f_{k'}\}_{k' \in \mathcal{K}}$ can be regarded as a p -adic analytic family of eigenforms of infinite $T(p)$ -slope passing through f in the sense of Definition 1.2.

Although the idea of the proof of this theorem given in this article can be guessed easily by any expert, it is worth while formulating a statement of p -adic analytic families of eigenforms of infinite $T(p)$ -slope.

Remark 1.3. (1) The family $\{a_n\}_{n \geq 1, (n,p)=1}$ obtained in the theorem above is a p -adic analytic interpolation of all Hecke eigensystems of the family $\{F_{k'}\}_{k' \in \mathcal{K}}$, i.e., the analytic function a_n gives us a p -adic analytic interpolation of eigenvalues of $F_{k'}$'s for the usual Hecke operator $T(n)$ for each integer $n \geq 1$ which is prime to p .

(2) Since ρ_f can be regarded as being associated to the twisted eigenform $f \otimes \mathbf{1}_p$ of level Np^2 , if the residual mod p representation associated to f is absolutely irreducible, then the p -adic analytic family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of eigenforms of infinite $T(p)$ -slope in the theorem above gives an affirmative answer to the part (2) of [17, Further Questions in Section 18] via the viewpoint of Remark 1.1.

(3) Note that some results on relations between (classical or overconvergent) eigenforms of infinite $T(p)$ -slope and of finite $T(p)$ -slopes have been known. For example, Coleman and Stein [9, Theorem 2.1] constructed a pointwise family of classical eigenforms of finite $T(p)$ -slopes converging to a twisted classical eigenform of infinite $T(p)$ -slope. On the other hand, Calegari [3, Theorem 1.1] proved by investigating the geometry of “the Coleman-Mazur eigencurve” (cf. [8]) that there exists a p -adic family of overconvergent eigenforms of finite $T(p)$ -slopes over a punctured disk which converges to an overconvergent eigenform of infinite $T(p)$ -slope at the puncture corresponding to a certain arithmetic weight.

Example 1.1. Let E be an elliptic curve defined over \mathbb{Q} which has good supersingular reduction at p with conductor N and $a_p(E) := p + 1 - \sharp E(\mathbb{F}_p) = 0$, where the symbol \sharp stands for the cardinality and \mathbb{F}_p is the finite field with p elements. By the modularity of elliptic curves defined over \mathbb{Q} , namely the Shimura-Taniyama conjecture proved by Wiles [22], Taylor-Wiles [21], Conrad-Diamond-Taylor [10] and Breuil-Conrad-Diamond-Taylor [2], there exists the normalized newform f_E of weight 2, conductor N and trivial character associated to E . In particular, f_E has infinite $T(p)$ -slope, since $a_p(E) = a_p(f) = 0$.

Therefore, by Theorem 1.1, there exists a p -adic analytic family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of eigenforms of infinite $T(p)$ -slope passing through $f_E (= f_2)$ parametrized by $\mathcal{K} := \{2 + t(p-1)p^m \mid t = 0, 1, 2, \dots\}$ with some integer $m \geq 0$, which has a p -adic analytic interpolation ρ of associated Galois representations.

The author would like to pursue his study on the problem that how these families of infinite $T(p)$ -slope contribute to the arithmetic of p -supersingular elliptic curves.

Remark 1.4. Let \mathbb{A} be the adèle ring over \mathbb{Q} . For the newform f taken as in Theorem 1.1, the local p -component $\pi(f)_p$ of the automorphic representation $\pi(f)$ of $\mathrm{GL}_2(\mathbb{A})$ generated by f is a principal series representation, since the conductor N is prime to p (we call this case *p -supersingular case* in Definition 4.1. See Proposition 4.1 (1)). Then the Jacquet module J_f associated to $\pi(f)_p$ does not vanish. The Hecke operator $T(p)$ acts on J_f as an automorphism and the p -stabilized newforms coming from f correspond to the $T(p)$ -eigenvectors in J_f of *finite* $T(p)$ -slope. In this way, we can use a Coleman family of finite $T(p)$ -slope to obtain a p -adic analytic family of eigenforms of infinite $T(p)$ -slope passing through f .

On the other hand, if the conductor of a newform g of infinite $T(p)$ -slope is divisible by p , there are three cases about the p -local representation $\pi(g)_p$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to g , namely, the p -ramified principal series case, the p -ramified special case and the p -supercuspidal case (Proposition 4.1 (2)). In the p -ramified principal series case and the p -ramified special case, as explained in Section 4, we can also obtain p -adic analytic families of eigenforms of infinite $T(p)$ -slope passing through g by the same argument as in the p -supersingular case.

However, in the *p -supercuspidal case*, since the associated Jacquet module J_g vanishes, there exists no eigenform of finite $T(p)$ -slope whose Galois representation is equivalent to ρ_g . Therefore, we cannot use any Coleman family of finite $T(p)$ -slope to construct p -adic analytic families of eigenforms of infinite $T(p)$ -slope. The author would like to pursue his study on the question

whether it is possible to construct p -adic analytic families of eigenforms of infinite $T(p)$ -slope passing through p -supercuspidal newforms or not.

In Section 2, we shall recall Coleman's construction given in [6] of a p -adic analytic family of eigenforms of any finite $T(p)$ -slope passing through a given eigenform which is new away from p . Especially, as mentioned in Remark 1.2, we shall give a detailed proof of [6, Corollary B5.7.1]. In Section 3, we shall prove Theorem 1.1. In Section 4, we shall make a remark on newforms of infinite $T(p)$ -slope concerning structures of associated representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and p -local l -adic Galois representations with $l \neq p$ from the viewpoint of the local Langlands correspondence.

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2. Coleman families of finite $T(p)$ -slope

In this section, we shall recall Coleman's construction given in [6] of p -adic analytic families of eigenforms of any finite $T(p)$ -slope passing through a given eigenform which is new away from p . These families are called *Coleman families*. We often use the same terminology and notation as in [6] without detailed explanation.

2.1. Overconvergent and classical cusp forms

Let p be an odd prime number, $N \geq 1$ integer prime to p and $k \geq 2$ an integer. Let f be a cuspidal normalized eigenform of level Np , weight k and character ε with $T(n)$ -eigenvalues $a_n(f)$ for any $n \geq 1$. (Note that, the conditions which the eigenform f satisfies in this section are completely different from the assumption in Theorem 1.1, although we use the same symbol f as in the theorem.) Then the Fourier expansion of f is given by

$$f(q) = \sum_{n \geq 1} a_n(f) q^n,$$

since f is normalized.

The Dirichlet character $\varepsilon : (\mathbb{Z}/Np\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}^\times$ can be regarded as taking values in $\bar{\mathbb{Q}}_p^\times$ via composing with the fixed embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We can decompose as $\varepsilon = \varepsilon_N \varepsilon_p$ with

the N -part $\varepsilon_N : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}_p^\times$ and the p -part $\varepsilon_p : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}_p^\times$. We call ε_p the p -character of f . Then there exists some $i \in \{0, 1, \dots, p-1\}$ such that we have $\varepsilon_p = \tau^{i-k}$, where $\tau : (\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times \hookrightarrow \bar{\mathbb{Q}}_p^\times$ is the Teichmüller character.

We assume that f is *new away from p* (i.e, the conductor of f is equal to Np or N) and that f has *finite $T(p)$ -slope α* with $\alpha < k-1$. Let $S(N, i)$ be the space of families of cuspidal overconvergent forms of tame level N and type i defined in [6, Section B4]. Then by [6, Theorem B3.4], there exists a sufficiently large integer $m(\alpha)$ depending on α such that we can obtain a certain direct summand $S(N, i)_B^\alpha$ of the restriction $S(N, i)_B$ of $S(N, i)$ on the affinoid disk B of radius $p^{-m(\alpha)}$ around k defined over \mathbb{C}_p , which interpolates the \mathbb{C}_p -vector spaces $S_{k'}^{\text{cl}}(Np, \tau^{i-k'})^\alpha$ of classical cusp forms of level Np , p -character $\tau^{i-k'}$ and $T(p)$ -slope α with varying integral weights $k' \in B(\mathbb{Z})$ greater than $\alpha + 1$. Here the classicality of overconvergent forms of small $T(p)$ -slope is given by [5, Theorem 6.1]. (Note that $m(\alpha)$ and $S(N, i)_B^\alpha$ are written as r and H in [6, the subsection “ R -families” on the page 465], respectively.) The set of \mathbb{C}_p -valued points of the affinoid disk B is given by

$$B(\mathbb{C}_p) = \{s \in \mathcal{O}_{\mathbb{C}_p} \mid |k - s| \leq p^{-m(\alpha)}\}.$$

We denote by $A(B)$ the affinoid algebra associated to B . We know that the \mathbb{C}_p -vector spaces $S_{k'}^{\text{cl}}(Np, \tau^{i-k'})^\alpha$ have the same dimension, which we denote by d , for all such k' 's by [6, Theorem B3.4]. Then we see that $S(N, i)_B^\alpha$ is a projective $A(B)$ -module of rank d by [6, Theorem A4.5], and for such any k' , we have the specialization map

$$\text{sp}_{k'} : S(N, i)_B^\alpha \rightarrow S(N, i)_B^\alpha \otimes_{A(B)} A(B)/P_{k'} \cong S_{k'}^{\text{cl}}(Np, \tau^{i-k'})^\alpha,$$

where $P_{k'}$ is the maximal ideal of $A(B)$ at the closed point k' in B . Let \mathcal{K} be the arithmetic progression of radius $(p-1)p^{m(\alpha)}$ starting from k . For any $k' \in \mathcal{K}$, we then see that $k' \in B(\mathbb{Z})$, $k' > \alpha + 1$ and $\tau^{i-k'} = \tau^{i-k} = \varepsilon_p$.

Definition 2.1. The (p) -new subspace $S(N, i)_B^{(p)\text{-new}, \alpha}$ of $S(N, i)_B^\alpha$ is defined as the intersection of kernels of all the degeneracy trace maps from level $\Gamma_1(Np)$ to level $\Gamma_1(N'p)$ for all positive divisors N' of N with $N' \neq N$. For any $k' \in \mathcal{K}$, for the space $S_{k'}^{\text{cl}}(Np, \varepsilon_p)^\alpha$ of classical cusp forms of $T(p)$ -slope α , we can define its (p) -new subspace $S_{k'}^{\text{cl}}(Np, \varepsilon_p)^{(p)\text{-new}, \alpha}$ as well.

We can show that taking the (p) -new subspaces commutes with specializing the space of overconvergent cusp forms to the spaces of classical cusp forms of $T(p)$ -slope α with weights $k' \in \mathcal{K}$. Namely, we have the following

Proposition 2.1. *For any $k' \in \mathcal{K}$, we have a canonical isomorphism*

$$S(N, i)_B^{(p)\text{-new}, \alpha} \otimes_{A(B)} A(B)/P_{k'} \cong S_{k'}^{\text{cl}}(Np, \varepsilon_p)^{(p)\text{-new}, \alpha}$$

of finite dimensional \mathbb{C}_p -vector spaces.

Proof. Let $S := S(N, i)_B^\alpha$, $S^{(p)\text{-new}} := S(N, i)_B^{(p)\text{-new}, \alpha}$ and $\{T_j : S \rightarrow S_j := S(N_j, i)_B\}_j$ be the set of all the degeneracy trace maps which are used to define the (p) -new subspaces in Definition 2.1 with suitable divisors N_j of N . Putting $\mathbb{C}_{p, k'} := A(B)/P_{k'}$ for any $k' \in \mathcal{K}$, we consider the

following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S^{(p)\text{-new}} & \xrightarrow{i} & S & \xrightarrow{\oplus_j T_j} & \oplus_j S_j \\
 & & \text{id} \otimes 1 \downarrow & & \text{id} \otimes 1 \downarrow & & \text{id} \otimes 1 \downarrow \\
 & & S^{(p)\text{-new}} \otimes_{A(B)} \mathbb{C}_{p,k'} & \xrightarrow{i \otimes \text{id}} & S \otimes_{A(B)} \mathbb{C}_{p,k'} & \xrightarrow{(\oplus_j T_j) \otimes \text{id}} & (\oplus_j S_j) \otimes_{A(B)} \mathbb{C}_{p,k'} \\
 & & & & & & \parallel \\
 & & & & & & \oplus_j (S_j \otimes_{A(B)} \mathbb{C}_{p,k'}),
 \end{array}$$

where the first horizontal sequence is an exact one induced by the definition of (p) -new subspaces $S^{(p)\text{-new}}$ of S . Since we have canonical isomorphisms

$$S \otimes_{A(B)} \mathbb{C}_{p,k'} \cong S_k^{\text{cl}}(Np, \varepsilon_p)^\alpha$$

and for any j ,

$$S_j \otimes_{A(B)} \mathbb{C}_{p,k'} \cong S_{k'}^{\text{cl}}(N_j p, \varepsilon_p)^\alpha,$$

it suffices to see that (i) the homomorphism

$$i \otimes \text{id} : S^{(p)\text{-new}} \otimes_{A(B)} \mathbb{C}_{p,k'} \rightarrow S \otimes_{A(B)} \mathbb{C}_{p,k'}$$

is injective and that (ii) we have

$$\text{Image}(i \otimes \text{id}) = \text{Ker}((\oplus_j T_j) \otimes \text{id}).$$

(i) Since $A(B)$ is a principal ideal domain, the projective $A(B)$ -module S of finite rank d is free over $A(B)$. For the $A(B)$ -submodule $S^{(p)\text{-new}}$ of S , we then obtain a free basis s_1, \dots, s_d of S over $A(B)$ such that we have

$$S^{(p)\text{-new}} = A(B)a_1 s_1 \oplus \dots \oplus A(B)a_r s_r$$

with some $r \leq d$ and suitable elements $a_1, \dots, a_r \in A(B)$. Since the quotient $S/S^{(p)\text{-new}}$ can be regarded as an $A(B)$ -submodule of the $A(B)$ -torsion free module $\oplus_j S_j$, we see that a_1, \dots, a_r are units in $A(B)$ and $S^{(p)\text{-new}}$ is a direct summand of S . Therefore $i \otimes \text{id}$ is an injection.

(ii) Since the second square in the above diagram

$$\begin{array}{ccc}
 S & \xrightarrow{\oplus_j T_j} & \oplus_j S_j \\
 \text{id} \otimes 1 \downarrow & & \text{id} \otimes 1 \downarrow \\
 S \otimes_{A(B)} \mathbb{C}_{p,k'} & \xrightarrow{(\oplus_j T_j) \otimes \text{id}} & \oplus_j (S_j \otimes_{A(B)} \mathbb{C}_{p,k'})
 \end{array}$$

is commutative and tensoring with $\mathbb{C}_{p,k'} = A(B)/P_{k'}$ is a right exact functor on $A(B)$ -modules, we then see that

$$\text{Image}(i \otimes \text{id}) = \text{Ker}((\oplus_j T_j) \otimes \text{id}).$$

Thus the proposition is proved. \square

Definition 2.2. For the initial weight k , we define the subspace $S_k^{\text{cl,ss}}$ of $S_k^{\text{cl}}(Np, \varepsilon_p)^{(p)\text{-new}, \alpha}$ as the subspace generated by newforms of conductor Np and old forms $g(q)$ and $g(q^p)$ coming from newforms g of conductor N such that the characteristic polynomials of $T(p)$ acting on the subspaces spanned by $g(q)$ and $g(q^p)$ has no double roots. Note that the superscript “ss” means that the Hecke algebra \mathfrak{h}_k acts semi-simply on the subspace $S_k^{\text{cl,ss}}$ as we shall see in the next subsection.

By Proposition 2.1, we have the specialization map

$$\begin{aligned} \mathrm{sp}_k : S(N, i)_B^{(p)\text{-new}, \alpha} &\xrightarrow{\mathrm{id} \otimes 1} S(N, i)_B^{(p)\text{-new}, \alpha} \otimes_{A(B)} A(B)/P_k \\ &\cong S_k^{\mathrm{cl}}(Np, \varepsilon_p)^{(p)\text{-new}, \alpha}. \end{aligned}$$

Then we put

$$S_B^{\mathrm{ss}} := \mathrm{sp}_k^{-1}(S_k^{\mathrm{cl}, \mathrm{ss}}) \subset S(N, i)_B^{(p)\text{-new}, \alpha}.$$

2.2. Hecke algebras and Coleman families

Since we assume that f is new away from p , we see that f belongs to $S_k^{\mathrm{cl}}(Np, \varepsilon_p)^{(p)\text{-new}, \alpha}$. Further, assuming that f is in the subspace $S_k^{\mathrm{cl}, \mathrm{ss}}$ defined in Definition 2.2, we shall see a detailed proof of the existence of Coleman family, i.e., a p -adic analytic family of eigenforms of $T(p)$ -slope α passing through f in this subsection.

Definition 2.3. The space $S(N, i)_B^\alpha$ is stable under the actions of Hecke operators $T(n)$ with all positive integers n defined in [6, Section B5]. We denote by \mathcal{H} the *Hecke algebra* defined as the $A(B)$ -subalgebra in $\mathrm{End}_{A(B)}(S(N, i)_B^\alpha)$ generated by Hecke operators $T(n)$ with all $n \geq 1$. (Note that \mathcal{H} is written as R in [6, the subsection “ R -families” on the page 465].) Then the subspace $S(N, i)_B^{(p)\text{-new}, \alpha}$ of $S(N, i)_B^\alpha$ is stable under the action of the Hecke algebra \mathcal{H} . We denote by $\mathcal{H}^{(p)\text{-new}}$ the image of the natural homomorphism

$$\mathcal{H} \rightarrow \mathrm{End}_{A(B)}(S(N, i)_B^{(p)\text{-new}, \alpha})$$

given by restricting the Hecke action. Since the $A(B)$ -submodule S_B^{ss} defined in Definition 2.2 is stable under the action of $\mathcal{H}^{(p)\text{-new}}$, we can take the image \mathfrak{h} of the natural homomorphism

$$\mathcal{H}^{(p)\text{-new}} \rightarrow \mathrm{End}_{A(B)}(S_B^{\mathrm{ss}})$$

given by restricting the Hecke action.

Then \mathfrak{h} is a \mathbb{C}_p -affinoid algebra which is finite over $A(B)$ because so is $\mathcal{H}^{(p)\text{-new}}$ as in *loc.cit.*. We specialize \mathfrak{h} at the closed point k of B as $\mathfrak{h} \otimes_{A(B)} A(B)/P_k$ and take the image \mathfrak{h}_k of the natural homomorphism

$$\mathfrak{h} \otimes_{A(B)} A(B)/P_k \rightarrow \mathrm{End}_{\mathbb{C}_p}(\mathrm{sp}_k(S_B^{\mathrm{ss}})) = \mathrm{End}_{\mathbb{C}_p}(S_k^{\mathrm{cl}, \mathrm{ss}}).$$

Then the Hecke algebra \mathfrak{h}_k is a reduced semi-simple \mathbb{C}_p -algebra by the theory of newforms and old forms (cf. [18, Theorem 1] and [15, Proposition 3.23]). By the definitions of \mathfrak{h} and \mathfrak{h}_k , we have the natural surjective $A(B)$ -algebra homomorphism

$$\varphi_k : \mathfrak{h} \rightarrow \mathfrak{h}_k.$$

Remark 2.1. Since we assume that f is in $S_k^{\mathrm{cl}, \mathrm{ss}}$, we see that $S_k^{\mathrm{cl}, \mathrm{ss}}$ is a non-zero \mathbb{C}_p -vector space and that \mathfrak{h}_k is a non-zero \mathbb{C}_p -algebra.

Let $\lambda_1, \dots, \lambda_r : \mathfrak{h}_k \rightarrow \mathbb{C}_p$ be \mathbb{C}_p -algebra homomorphisms which correspond to all \mathfrak{h}_k -eigensystems on $S_k^{\mathrm{cl}, \mathrm{ss}}$ with some positive integer $r \leq d$. We may assume that λ_1 corresponds to the system of \mathfrak{h}_k -eigenvalues of the eigenform f . We know that the semi-simple \mathbb{C}_p -algebra \mathfrak{h}_k can be decomposed as

$$\begin{aligned} \mathfrak{h}_k &\xrightarrow{\sim} \mathbb{C}_p \times \mathbb{C}_p \times \cdots \times \mathbb{C}_p, \\ T &\mapsto (\lambda_1(T), \lambda_2(T), \dots, \lambda_r(T)) \end{aligned}$$

by the strong multiplicity one theorem, since the Hecke algebra \mathfrak{h}_k contains the Hecke operator $T(p)$ at p .

Let $\mathfrak{h}_{\text{red}} := \mathfrak{h}/\sqrt{(0)}$ be the reduction of \mathfrak{h} . Since \mathfrak{h}_k is reduced, we then see that φ_k factors through the $A(B)$ -algebra surjection

$$\varphi_k : \mathfrak{h}_{\text{red}} \rightarrow \mathfrak{h}_k.$$

Now we can show that the reduced Hecke algebra $\mathfrak{h}_{\text{red}}$ has a decomposition given by lifting the decomposition of \mathfrak{h}_k to $\mathfrak{h}_{\text{red}}$. Namely, we obtain the following

Theorem 2.2. *We have the following commutative diagram of $A(B)$ -algebras*

$$\begin{array}{ccc} \mathfrak{h}_{\text{red}} & \xrightarrow{\sim} & A(B) \times A(B) \times \cdots \times A(B), \quad T \mapsto (A_T^{(1)}, A_T^{(2)}, \dots, A_T^{(r)}) \\ \varphi_k \downarrow & & \downarrow \text{mod } P_k \\ \mathfrak{h}_k & \xrightarrow{\sim} & \mathbb{C}_p \times \mathbb{C}_p \times \cdots \times \mathbb{C}_p, \quad T \mapsto (\lambda_1(T), \lambda_2(T), \dots, \lambda_r(T)) \end{array}$$

with shrinking the disk B around the center k if necessary.

In particular, we have for any $T \in \mathfrak{h}_{\text{red}}$ and $i = 1, 2, \dots, r$,

$$A_T^{(i)} \pmod{P_k} = \lambda_i(\varphi_k(T)).$$

Proof. Let $\mathfrak{h}_{\text{red},(k)} := \mathfrak{h}_{\text{red}} \otimes_{A(B)} A(B)_{P_k}$ be the localization of $\mathfrak{h}_{\text{red}}$ at k . Since the localization $A(B)_{P_k}$ of the affinoid algebra $A(B)$ at the maximal ideal P_k is Henselian and $\mathfrak{h}_{\text{red},(k)}$ is finite over $A(B)_{P_k}$, we have the following commutative diagram of decompositions of $A(B)_{P_k}$ -algebras

$$\begin{array}{ccc} \mathfrak{h}_{\text{red},(k)} & \xrightarrow{\sim} & \mathfrak{h}_{\text{red},1} \times \mathfrak{h}_{\text{red},2} \times \cdots \times \mathfrak{h}_{\text{red},r} \\ \varphi_k \downarrow & & \downarrow \text{mod } P_k \\ \mathfrak{h}_k & \xrightarrow{\sim} & \mathbb{C}_p \times \mathbb{C}_p \times \cdots \times \mathbb{C}_p \end{array}$$

with bijection between the respective idempotents of $\mathfrak{h}_{\text{red},(k)}$ and \mathfrak{h}_k inducing the above decompositions by [20, Propositions I.3 and I.4]. Since $A(B)_{P_k}$ is flat over $A(B)$, we can decompose φ_k as

$$\varphi_k : \mathfrak{h}_{\text{red},(k)} \xrightarrow{\text{mod } P_k} \mathfrak{h}_{\text{red},(k)} / P_k \mathfrak{h}_{\text{red},(k)} \xrightarrow{\psi} \mathfrak{h}_k$$

with natural surjection ψ . By Proposition 2.1 and [4, Lemme 6.2.5], we then see that $\text{Ker}(\psi)$ is nilpotent. Therefore ψ is an isomorphism.

Since the space S_B^{ss} is $A(B)$ -torsion free, $A(B)$ is reduced and $A(B)_{P_k}$ is flat over $A(B)$, the structure homomorphism

$$\iota : A(B)_{P_k} \rightarrow \mathfrak{h}_{\text{red},(k)}$$

is injective. In the decomposition of $\mathfrak{h}_{\text{red},(k)}$ above, $A(B)_{P_k}$ is embedded by ι diagonally as

$$\begin{array}{ccccc} A(B)_{P_k} & \xhookrightarrow{\iota} & \mathfrak{h}_{\text{red},(k)} & \xrightarrow{\sim} & \mathfrak{h}_{\text{red},1} \times \mathfrak{h}_{\text{red},2} \times \cdots \times \mathfrak{h}_{\text{red},r} \\ \text{mod } P_k \downarrow & & \downarrow \varphi_k & & \downarrow \text{mod } P_k \\ \mathbb{C}_p & \hookrightarrow & \mathfrak{h}_k & \xrightarrow{\sim} & \mathbb{C}_p \times \mathbb{C}_p \times \cdots \times \mathbb{C}_p. \end{array}$$

Since $\psi : \mathfrak{h}_{\text{red},(k)}/P_k \mathfrak{h}_{\text{red},(k)} \xrightarrow{\sim} \mathfrak{h}_k$ is an isomorphism as we have seen above, we have the following commutative diagram at the i -th component for each $i = 1, 2, \dots, r$

$$\begin{array}{ccc} A(B)_{P_k} & \hookrightarrow & \mathfrak{h}_{\text{red},i} \\ \text{mod } P_k \downarrow & & \downarrow \text{mod } P_k \\ \mathbb{C}_p & \xrightarrow{\text{id}} & \mathbb{C}_p, \end{array}$$

where vertical homomorphisms are surjective. Then, by Nakayama's lemma, the injection $A(B)_{P_k} \hookrightarrow \mathfrak{h}_{\text{red},i}$ is an isomorphism for each $i = 1, 2, \dots, r$. Therefore we can obtain the desired commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_{\text{red}} & \xrightarrow{\sim} & A(B) \times A(B) \times \cdots \times A(B), \quad T \mapsto (A_T^{(1)}, A_T^{(2)}, \dots, A_T^{(r)}) \\ \varphi_k \downarrow & & \downarrow \text{mod } P_k \\ \mathfrak{h}_k & \xrightarrow{\sim} & \mathbb{C}_p \times \mathbb{C}_p \times \cdots \times \mathbb{C}_p, \quad T \mapsto (\lambda_1(T), \lambda_2(T), \dots, \lambda_r(T)) \end{array}$$

with shrinking the disk B around the center k if necessary. \square

Since we assume that λ_1 corresponds to the system of \mathfrak{h}_k -eigenvalues of the eigenform f , we can see the existence of the Coleman family passing through f as in the following

Corollary 2.3. *For any eigenform f of weight k , level Np and character ε which is in $S_k^{\text{cl,ss}}$ with $T(p)$ -slope $\alpha < k - 1$, there exists a p -adic analytic family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of cuspidal normalized eigenforms $f_{k'}$ of weight k' , level Np and character ε having $T(p)$ -slope α passing through f parametrized by the arithmetic progression \mathcal{K} of radius $(p - 1)p^m$ starting from k with some integer $m \geq 0$, which has the following properties:*

(1) ([6, Corollary B5.7.1]) *The Fourier expansions of $f_{k'}$'s for all $k' \in \mathcal{K}$ are interpolated p -adically by the formal power series*

$$F(q) := \sum_{n \geq 1} A_{T(n)}^{(1)} q^n$$

in the indeterminate q with the family $\{A_{T(n)}^{(1)}\}_{n \geq 1} \subset A(B)$ of analytic functions obtained in Theorem 2.2, i.e., we can obtain the Fourier expansion of $f_{k'}$ by

$$f_{k'}(q) = \sum_{n \geq 1} A_{T(n)}^{(1)}(k') q^n$$

for each $k' \in \mathcal{K}$;

(2) *By shrinking the affinoid disk B around k if necessary, we have an irreducible Galois representation*

$$\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(A(B))$$

unramified outside $S_{Np} := \{\text{the prime divisors of } Np\} \cup \{\infty\}$ such that for any prime number $l \notin S_{Np}$, we have

$$\text{Trace}(\rho(\text{Frob}_l)) = A_{T(l)}^{(1)},$$

where Frob_l is the geometric Frobenius element at l .

Therefore we see by (1) that for each $k' \in \mathcal{K}$,

$$\rho \pmod{P_{k'}} \cong \rho_{f_{k'}}.$$

Proof. For the first component of the decomposition in Theorem 2.2, we see that

$$A_{T(n)}^{(1)}(k) = A_{T(n)}^{(1)} \pmod{P_k} = \lambda_1(\varphi_k(T(n)))$$

is equal to the $T(n)$ -eigenvalue $a_n(f)$ of f for any $n \geq 1$. Therefore we have

$$f_k(q) = \sum_{n \geq 1} A_{T(n)}^{(1)}(k) q^n = \sum_{n \geq 1} a_n(f) q^n = f(q).$$

Moreover, for any $k' \in \mathcal{K}$, by the duality between classical eigenforms and \mathbb{C}_p -algebra homomorphisms from classical Hecke algebras to \mathbb{C}_p (cf. [15, Proposition 3.21]), we can see that $\{A_{T(n)}^{(1)}(k')\}_{n \geq 1}$ gives the Hecke eigensystem of some eigenform $f_{k'}$ belonging to $S_{k'}^{\text{cl}}(Np, \varepsilon_p)^{(p)\text{-new}, \alpha}$. By enlarging the integer m if necessary, we can fix the N -part of the characters of all $f_{k'}$'s as ε_N by [1, Lemma 5.5]. Thus the part (1) of the corollary is proved.

In order to prove the part (2) of the corollary, we shall use the pseudo-representation theory under the assumption that p is odd. For each $k' = k + i(p-1)p^m \in \mathcal{K}$ ($i = 0, 1, 2, \dots$), we have a continuous irreducible Galois representation

$$\rho_i := \rho_{f_{k'}} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{C}_p)$$

unramified outside S_{Np} which is associated to the eigenform $f_{k'}$. Then we have the continuous pseudo-representation

$$\phi_i = (a_i, d_i, x_i) : G_{\mathbb{Q}} \rightarrow \mathbb{C}_p$$

associated to ρ_i with fixed basis of the representation space $V_i = \mathbb{C}_p \times \mathbb{C}_p$ of ρ_i such that $\rho_i(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, where $c \in G_{\mathbb{Q}}$ is the complex conjugation (cf. [14, Section 7.5]). Then we have

$$(\text{Trace}(\phi_i))(\text{Frob}_l) = A_{T(l)}^{(1)}(k') = A_{T(l)}^{(1)} \pmod{P_i}$$

for any prime number $l \notin S_{Np}$, where we put $P_i := P_{k'}$. For each $i = 0, 1, 2, \dots$, we now define a continuous pseudo-representation

$$\Phi_i : G_{\mathbb{Q}} \rightarrow A(B)/P_0 \cap \dots \cap P_i$$

via the traces

$$(\text{Trace}(\Phi_i))(\text{Frob}_l) := A_{T(l)}^{(1)} \pmod{P_0 \cap \dots \cap P_i}$$

at the geometric Frobenius elements $\{\text{Frob}_l\}_{l \notin S_{Np}}$ by Chebotarev density theorem. Since the family $\{P_i\}_{i=0}^{\infty}$ of maximal ideals of $A(B)$ satisfies that

$$A(B) \xrightarrow{\sim} \varprojlim_i A(B)/P_0 \cap \dots \cap P_i$$

by Weierstrass preparation theorem, we then have a continuous pseudo-representation

$$\Phi = (a, d, x) : G_{\mathbb{Q}} \rightarrow A(B)$$

as the inverse limit of Φ_i 's such that

$$\Phi \pmod{P_i} = \phi_i$$

for each $i = 0, 1, 2, \dots$ by [14, Proposition 7.5.2]. Since $\rho_0 (= \rho_k)$ is irreducible, there exist elements σ and τ in $G_{\mathbb{Q}}$ such that

$$0 \neq x_0(\sigma, \tau) = x(\sigma, \tau) \pmod{P_0}.$$

Therefore, by shrinking the affinoid disk B around k sufficiently to get it into the support of the analytic function $x(\sigma, \tau)$ if necessary, we can construct an irreducible Galois representation

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(A(B))$$

such that

$$\rho \pmod{P_i} \cong \rho_i$$

for each $i = 0, 1, 2, \dots$ (cf. [14, Proposition 7.5.1]). Since ρ_i is unramified outside S_{Np} for each $i = 0, 1, 2, \dots$, and the natural homomorphism

$$A(B) \rightarrow \prod_i A(B)/P_i$$

is injective, we see that ρ is also unramified outside S_{Np} and the part (2) of the corollary is proved. \square

3. p -Adic analytic families of infinite $T(p)$ -slope obtained by twisting Coleman families

In this section, we shall prove Theorem 1.1.

Let f be a cuspidal normalized newform of weight k , prime-to- p conductor N and character ε with $a_p(f) = 0$. Let f_+^* and f_-^* be the p -stabilized newforms of level Np , weight k and character ε coming from f defined by

$$\begin{aligned} f_+^*(q) &= f(q) - \sqrt{-\varepsilon(p)p^{k-1}}f(q^p) \quad \text{and} \\ f_-^*(q) &= f(q) + \sqrt{-\varepsilon(p)p^{k-1}}f(q^p), \end{aligned}$$

respectively. Then the eigenforms f_+^* and f_-^* are new away from p both with the $T(l)$ -eigenvalue $a_l(f)$ for any prime number l which is different from p , and $T(p)$ -eigenvalue $\sqrt{-\varepsilon(p)p^{k-1}}$ and $-\sqrt{-\varepsilon(p)p^{k-1}}$, respectively. Thus by virtue of Corollary 2.3 (1), we obtain a family $\{a_n^{(+)}\}_{n \geq 1}$ (resp. $\{a_n^{(-)}\}_{n \geq 1}$) of analytic functions on the 1-dimensional affinoid disk B of radius p^{-m} around k defined over \mathbb{C}_p with some integer $m \geq 0$ such that the formal power series

$$F_+(q) := \sum_{n \geq 1} a_n^{(+)} q^n \quad (\text{resp. } F_-(q) := \sum_{n \geq 1} a_n^{(-)} q^n)$$

in the indeterminate q gives Fourier expansions of all members of a p -adic analytic family $\{f_{k'}^{(+)}\}_{k' \in \mathcal{K}}$ (resp. $\{f_{k'}^{(-)}\}_{k' \in \mathcal{K}}$) of eigenforms $f_{k'}^{(+)}$ (resp. $f_{k'}^{(-)}$) of level Np , weight k' , character ε and $T(p)$ -slope $\frac{k-1}{2}$ passing through f_+^* (resp. f_-^*). Namely, we have

$$f_{k'}^{(+)}(q) = \sum_{n \geq 1} a_n^{(+)}(k') q^n \quad (\text{resp. } f_{k'}^{(-)}(q) = \sum_{n \geq 1} a_n^{(-)}(k') q^n)$$

for each $k' \in \mathcal{K}$ and $f_k^{(+)} = f_+^*$ (resp. $f_k^{(-)} = f_-^*$), where $\mathcal{K} = \{k' = k + i(p-1)p^m \mid i = 0, 1, 2, \dots\} \subset B(\mathbb{Z})$.

Paying attention to the plus objects above and putting $a_n := a_n^{(+)}$ for any $n \geq 1$ which is prime to p , we then see that the formal power series

$$F(q) := \sum_{n \geq 1, (n, p) = 1} a_n q^n$$

in the indeterminate q gives Fourier expansions of all members of the family $\{F_{k'}\}_{k' \in \mathcal{K}}$ of cusp forms $F_{k'}$ of weight k' defined by for any $k' \in \mathcal{K}$,

$$F_{k'}(q) := \sum_{n \geq 1, (n, p) = 1} a_n(k') q^n.$$

Here $F_{k'}$ is a normalized eigenform of level Np^2 , character ε and infinite $T(p)$ -slope for any $k' \in \mathcal{K}$ by [19, Lemma 4.6.5]. In particular, the specialization F_k at k is identified with the twisted eigenform $f_+^* \otimes \mathbf{1}_p$.

Moreover, by Corollary 2.3 (2), we also have an irreducible Galois representation

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(A(B))$$

unramified outside S_{Np} such that for any prime number $l \notin S_{Np}$, we have

$$\mathrm{Trace}(\rho(\mathrm{Frob}_l)) = a_l^{(+)} = a_l.$$

Therefore, for each $k' \in \mathcal{K}$, we have

$$\rho \pmod{P_{k'}} \cong \rho_{F_{k'}}.$$

Since the specialization ρ_{F_k} at k is equivalent to the Galois representation ρ_f associated to f , putting $f_k := f$ and $f_{k'} := F_{k'}$ for any $k' \in \mathcal{K} \setminus \{k\}$, we then see that the family $\{f_{k'}\}_{k' \in \mathcal{K}}$ is a p -adic analytic family of eigenforms of infinite $T(p)$ -slope passing through f in the sense of Definition 1.2. Thus Theorem 1.1 is proved.

Remark 3.1. (1) We can obtain another twisted formal power series

$$F'(q) := \sum_{n \geq 1, (n, p) = 1} a_n^{(-)} q^n$$

with the minus objects which has the same properties as $F(q)$ above to prove Theorem 1.1. We do not know if $F(q)$ and $F'(q)$ are equal or not, although their specialization at k are equal. This problem seems to be connected with the ramification property at k of the reduced p -adic Hecke algebra $\mathfrak{h}_{\mathrm{red}}$ in Theorem 2.2.

(2) When the residual mod p representation $\bar{\rho}$ associated to f is absolutely irreducible, if the two p -adic analytic families $\{f_{k'}^{(+)}\}_{k' \in \mathcal{K}}$ and $\{f_{k'}^{(-)}\}_{k' \in \mathcal{K}}$ of eigenforms of the same $T(p)$ -slope $\frac{k-1}{2}$ are distinct, then they give the two distinct modular arcs in the universal deformation space for $\bar{\rho}$ intersecting at the only one point corresponding to ρ_f at the initial weight k , since the Galois representations associated to the p -stabilized newforms f_+^* and f_-^* are equivalent to ρ_f (cf. [17, Lemma 1 in Section 17]). On the contrary to [17, Lemma 2 in Section 17] under the assumption that the finite $T(p)$ -slopes are *distinct*, our families $\{f_{k'}^{(+)}\}_{k' \in \mathcal{K}}$ and $\{f_{k'}^{(-)}\}_{k' \in \mathcal{K}}$ give an affirmative evidence of [17, Question in Section 17] in the case of the *same* finite $T(p)$ -slope.

4. A representation theoretic remark on newforms of infinite $T(p)$ -slope

Before finishing this article, we shall make a remark on newforms of infinite $T(p)$ -slope in this section. Now we see a well-known fact about such newforms concerning the representation theory as

Proposition 4.1. *Let f be a cuspidal normalized newform of weight $k \geq 2$, conductor M and character ε having infinite $T(p)$ -slope. Let $c(\varepsilon)$ be the conductor of the Dirichlet character ε modulo M . We denote by M_p and $c(\varepsilon)_p$ the p -parts of M and $c(\varepsilon)$, respectively. Let $\pi(f)_p$ be*

the local p -component of the automorphic representation $\pi(f)$ of $\mathrm{GL}_2(\mathbb{A})$ generated by f , and D_p the decomposition group at p in $G_{\mathbb{Q}}$.

(1) If M is prime to p , then $\pi(f)_p$ is the principal series representation $\pi(\chi_-, \chi_+)$ given by the unramified characters $\chi_{\pm} : \mathbb{Q}_p^{\times} \rightarrow \bar{\mathbb{Q}}^{\times}$ defined as $\chi_{\pm}(p) := \pm \sqrt{-\varepsilon(p)p^{k-1}}$. In this case, the p -local l -adic Galois representation $\rho_{f,l} : D_p \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ associated to f with any prime number l distinct from p is equivalent to the diagonal form $\begin{pmatrix} \chi_+ & 0 \\ 0 & \chi_- \end{pmatrix}$, where χ_{\pm} are regarded as the p -local Galois character inducing the characters χ_{\pm} on \mathbb{Q}_p^{\times} via the local class field theory.

(2) If M is divisible by p , then we see that $p^2|M$ and $c(\varepsilon)_p \neq M_p$ and that there are the following three cases:

(i) If $\pi(f)_p$ is a principal series representation $\pi(\chi_1, \chi_2)$ with some ramified characters χ_1 and χ_2 on \mathbb{Q}_p^{\times} , then $\rho_{f,l} : D_p \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ is equivalent to the diagonal form $\begin{pmatrix} \chi_2 & 0 \\ 0 & \chi_1 \end{pmatrix}$, where χ_1 (resp. χ_2) is regarded as the p -local Galois character inducing the character χ_1 (resp. χ_2) on \mathbb{Q}_p^{\times} via the local class field theory.

(ii) If $\pi(f)_p$ is a special representation $\pi(\chi, \chi|\cdot|^{-1})$ with some ramified character χ on \mathbb{Q}_p^{\times} , then $\rho_{f,l} : D_p \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ is equivalent to the upper triangle form $\begin{pmatrix} \chi\chi_p & * \\ 0 & \chi \end{pmatrix}$, where χ is regarded as the p -local Galois character inducing the character χ on \mathbb{Q}_p^{\times} via the local class field theory, and χ_p is the cyclotomic character on D_p .

(iii) If $\pi(f)_p$ is a supercuspidal representation, then $\rho_{f,l} : D_p \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ is irreducible.

Proof. (1) Since we assume that M is prime to p , we see that $\pi(f)_p$ is a principal series representation $\pi(\chi_1, \chi_2)$ with suitable unramified characters $\chi_1, \chi_2 : \mathbb{Q}_p^{\times} \rightarrow \bar{\mathbb{Q}}^{\times}$. In this case, the p -local l -adic Galois representation $\rho_{f,l} : D_p \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ associated to f is equivalent to the upper triangle form $\begin{pmatrix} \chi_2 & * \\ 0 & \chi_1 \end{pmatrix}$ by [16, Section 2.3.8]. Here, by the local class field theory, we regard the character χ_i as being l -adic continuous Galois character on D_p induced by continuity from the character on the Weil group $W_{\mathbb{Q}_p}$ of \mathbb{Q}_p

$$W_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Q}_p^{\times} \xrightarrow{\chi_i} \bar{\mathbb{Q}}^{\times} \xrightarrow{i_l} \bar{\mathbb{Q}}_l^{\times}$$

for each $i = 1, 2$. Since f has infinite $T(p)$ -slope, the characteristic polynomial of $\rho_{f,i}(\mathrm{Frob}_p)$ is given by $X^2 + \varepsilon(p)p^{k-1}$ whose roots are $\pm \sqrt{-\varepsilon(p)p^{k-1}}$. This means that we have $\chi_2(p) + \chi_1(p) = 0$ and $\chi_2(p)\chi_1(p) = -\varepsilon(p)p^{k-1}$. Then we may have that

$$\begin{aligned} \chi_2(p) &= \sqrt{-\varepsilon(p)p^{k-1}} = \chi_+(p) \quad \text{and} \\ \chi_1(p) &= -\sqrt{-\varepsilon(p)p^{k-1}} = \chi_-(p). \end{aligned}$$

Since we then see that $(\chi_- \chi_+^{-1})(p) = -1 \neq |p|^{-1} = p$, $\pi(f)_p$ is not a special representation and $\rho_{f,l} : D_p \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ is equivalent to the diagonal form $\begin{pmatrix} \chi_+ & 0 \\ 0 & \chi_- \end{pmatrix}$ by *loc.cit.*

(2) If M is divisible by p , we see that $p^2|M$ and $c(\varepsilon)_p \neq M_p$ by [19, Theorem 4.6.17(3)], since the newform f has infinite $T(p)$ -slope. In this case, it is well-known that the representation theoretic classification of the forms of associated p -local l -adic Galois representations is given as in the statements via the local Langlands correspondence (cf. [16, Proposition 2.44]). \square

Definition 4.1. For a newform f of infinite $T(p)$ -slope, we say that f is p -supersingular if f is in the case of Proposition 4.1 (1). On the other hand, we say that f is p -supercuspidal if f is in the case of Proposition 4.1 (2) (iii), since the local p -component $\pi(f)_p$ of the automorphic representation $\pi(f)$ of $\mathrm{GL}_2(\mathbb{A})$ generated by f is a supercuspidal representation.

For example, if the newform f of conductor M having infinite $T(p)$ -slope is of weight 2 and associated to some elliptic curve of conductor M defined over \mathbb{Q} which has good supersingular reduction at p , then f is p -supersingular.

Theorem 1.1 insists that we can construct p -adic analytic families of eigenforms of infinite $T(p)$ -slope passing through f (of weight k) in the p -supersingular case. As we have seen in Section 3, the family of infinite $T(p)$ -slope obtained in Theorem 1.1 is constructed by twisting a certain Coleman family of $T(p)$ -slope $\frac{k-1}{2}$ passing through the plus p -stabilized newform f_+^* coming from f by the trivial Dirichlet character modulo p . Note that the two p -stabilized newforms f_\pm^* correspond to $T(p)$ -eigenvectors of $T(p)$ -slope $\frac{k-1}{2}$ in the Jacquet module associated to the principal series representation $\pi(\chi_-, \chi_+)$ given in Proposition 4.1 (1).

In the case of Proposition 4.1 (2) (i) (resp. (ii)), we can also obtain a $T(p)$ -eigenvector of $T(p)$ -slope $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) in the Jacquet module associated to $\pi(f)_p$. Then we can construct a p -adic analytic family of eigenforms of infinite $T(p)$ -slope passing through f as well as Theorem 1.1, since Coleman families of finite $T(p)$ -slope passing through a given (p) -new eigenform can be constructed in the higher level case by the same argument as in Section 2 using Coleman's theory of families of overconvergent forms of higher level (cf. [7]). Note that the $T(p)$ -slope calculation above in the case (i) (resp. (ii)) is given by [13, Lemma 12.2 (12.1a) (resp. (12.1b))], since the \mathbb{C}^\times -valued character χ_1 (resp. χ) in the statement is continuous on \mathbb{Q}_p^\times (in which \mathbb{Z}_p^\times is a profinite subgroup) and turned into an unramified character after being twisted by some character of finite order.

Finally, in the problem of constructing p -adic analytic families of eigenforms of infinite $T(p)$ -slope passing through a given newform f , the p -supercuspidal case (iii) is the most essential among all cases appeared in Proposition 4.1, since the associated Jacquet module vanishes and then we cannot use any Coleman family of finite $T(p)$ -slope to construct such a family with twisting argument as in Theorem 1.1 (cf. Remark 1.4).

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p -超特異的な場合における傾き無限大の固有形式からなる p -進解析的な無限族について

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要 旨

p を奇素数とし, f を重さ k で $T(p)$ -傾きが無限大, つまり $f|T(p) = 0$ であるような新形式とする。 f の導手 N は p と互いに素であると仮定する (このような場合を p -超特異的な場合という)。 f から来るレベル Np の p -安定化された新形式を f^* とすると, f^* の $T(p)$ -傾きは $(k-1)/2$ である。この論文では, f^* を通り $T(p)$ -傾きが $(k-1)/2$ のコールマン無限族を, p を法とする自明なディリクレ指標で捻ることで, f を通り $T(p)$ -傾きが無限大の固有形式からなる p -進解析的な無限族を構成する。

キーワード: 保型形式, ヘッケ環, ガロア表現, p -進無限族, 無限大の $T(p)$ -傾き