Real cross section of the connectedness locus of the family of polynomials $(z^{2n+1} + a)^{2n+1} + b$

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Abstract

Yeshun Sun & Yongcheng Yin [3] and H. Ishida & T. Itoh [2] presented a precise description of the real cross section of the connectedness locus of the family of bi-quadratic polynomials $\{(z^2+a)^2+b\}$. In this note, we shall give a precise description of the real cross section of the connectedness locus of the family of polynomials $\{(P_{2n+1,b} \circ P_{2n+1,a})(z)\} = \{(z^{2n+1}+a)^{2n+1}+b\}$, where a, b are complex numbers and n is a positive integer. Our proof is an elementary one.

Keywords: complex dynamics, real cross section, connectedness locus, polynomials, Julia set

1. Introduction and main results

Let $\{(P_{2n+1,b} \circ P_{2n+1,a})(z) = (z^{2n+1} + a)^{2n+1} + b\}$ be the family of polynomials with complex parameters a, b, where n is a fixed positive integer. The connectedness locus of the family $\{P_{2n+1,b} \circ P_{2n+1,a}\}$ is the set

$$C_{2n+1,\mathbb{C}} = \{(a,b) \in \mathbb{C}^2 \mid \text{Julia set of } P_{2n+1,b} \circ P_{2n+1,a} \text{ is connected.}\}$$

and the real cross section of $C_{2n+1,\mathbb{C}}$ is the set

$$C_{2n+1,\mathbb{R}} = \{(a,b) \in \mathbb{R}^2 \mid (a,b) \in C_{2n+1,\mathbb{C}}\}.$$

We shall prove the following

Theorem 1. $C_{2n+1,\mathbb{R}}$ *is the bounded closed region whose boundary is a simple closed curve consisting of two smooth pieces*

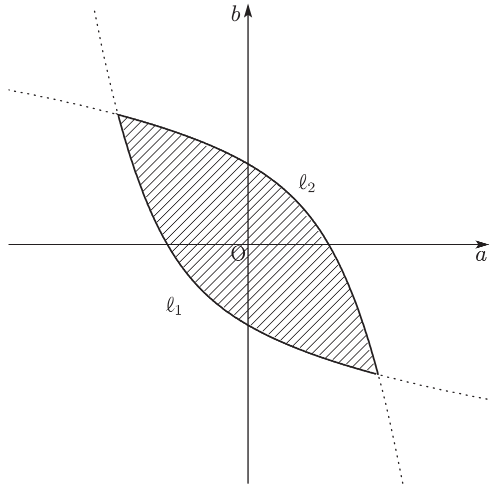
$$\ell_1 = \{(a,b) \in \mathbb{R}^2 \mid a = \frac{1}{\kappa t} - t^{2n+1},\ \ b = t - \frac{1}{(\kappa t)^{2n+1}}\ (-t_2 \leqq t \leqq -t_1)\},$$

$$\ell_2 = \{(a,b) \in \mathbb{R}^2 \mid a = \frac{1}{\kappa t} - t^{2n+1},\ \ b = t - \frac{1}{(\kappa t)^{2n+1}}\ (t_1 \leqq t \leqq t_2)\},$$

where $t_1 = \dfrac{1}{\sqrt{\alpha\kappa}}$, $t_2 = \dfrac{\sqrt{\alpha}}{\sqrt{\kappa}}$, $\kappa = \sqrt[n]{2n+1}$ *, and* $-\alpha$ *is a unique solution of the equation*

$$\sum_{j=-n}^{n} u^j = (-1)^n(2n+1)$$

satisfying $u < -1$.



2. Preliminaries

Let p be a polynomial whose degree is more than one. We denote the k-times iteration of p by p^k. A critical point c of p is a zero of the derivative p' of p, that is, $p'(c) = 0$, and a critical value of p is the image $p(c)$ of a critical point c of p. For a critical point c of p, $\{p^k(c)\}_{k=1}^{\infty}$ is a critical orbit of p.

We use the following well known fact. (See [1].)

Proposition 2. *The Julia set of p is connected if and only if all critical orbits of p are bounded.*

Here we note that $P_{2n+1,b} \circ P_{2n+1,a}$ has $2n+2$ critical points $0, -a^{1/(2n+1)}$. Since $(P_{2n+1,b} \circ P_{2n+1,a})(-a^{1/(2n+1)}) = b$, $P_{2n+1,b} \circ P_{2n+1,a}$ has only two critical orbits. Further, these critical orbits of $P_{2n+1,b} \circ P_{2n+1,a}$ are sequences of real numbers, *if both a and b are real numbers*.

3. Proof of Theorem 1.

For simplicity, we denote $(P_{2n+1,b} \circ P_{2n+1,a})(z) = (z^{2n+1}+a)^{2n+1}+b$ by $P(z)$. Set $F(z) = P(z)-z$ then a fixed point of P is a solution of the equation $F(z) = 0$. Since

$$P'(z) = (2n+1)^2 z^{2n}(z^{2n+1}+a)^{2n},$$

we have

$$F'(z) = P'(z) - 1 = (2n+1)^2 z^{2n}(z^{2n+1}+a)^{2n} - 1$$
$$= h(z)(h(z)+2),$$

where

$$h(z) = (2n+1)z^n(z^{2n+1}+a)^n - 1.$$

Hereafter, we assume that *both a and b be real parameters*. Let denote the real part of the complex variable z by x.

$P'(x) = 0$ has two real solutions $0, -\sqrt[2n+1]{a}$. Let

$$c_1 = \min\{0, -\sqrt[2n+1]{a}\}$$
$$c_2 = \max\{0, -\sqrt[2n+1]{a}\},$$

then $c_1 \leqq c_2$ and the equality holds when $a = 0$.

Since the degree of $F(x)$ is $(2n+1)^2$, $F(x) = 0$ has at least one real solution. Denote r_{min} and r_{max} by the least and the greatest real solution of $F(x) = 0$, that is, the least and the greatest real fixed point of P respectively.

We shall prove the following three lemmas.

Lemma 3. *Both critical orbits $\{P^k(c_1)\}_k$ and $\{P^k(c_2)\}_k$ are bounded, that is,*

$$(a,b) \in C_{2n+1,\mathbb{R}}$$

if and only if $F(x) = P(x) - x = 0$ has at least two real solutions and

$$c_1, c_2 \in [r_{min}, r_{max}] = \{x \,|\, r_{min} \leqq x \leqq r_{max}\}.$$

Proof.

First, assume that $F(x) = P(x) - x = 0$ has only one real solution $x = r$. Then $P'(r) \geqq 1$. Since $P'(c_1) = P'(c_2) = 0$, both c_1 and c_2 are not equal to r. Hence, either $c_1 < r$ or $c_2 > r$. Note that $P(x) < x$ if $x < r$ and $P(x) > x$ if $x > r$. Then, if $c_1 < r$, $\lim_{k\to\infty} P^k(c_1) = -\infty$. Similarly, if $c_2 > r$, $\lim_{k\to\infty} P^k(c_2) = \infty$.

Thus, if both critical orbits $\{P^k(c_1)\}$ and $\{P^k(c_2)\}$ are bounded, then $F(x) = P(x) - x = 0$ must have at least two real solutions.

Moreover, $P(x) < x$ when $x < r_{min}$ and $P(x) > x$ when $x > r_{max}$. Hence, if $c_1 < r_{min}$, then $\lim_{k\to\infty} P^k(c_1) = -\infty$. Similarly, if $c_2 > r_{max}$, then $\lim_{k\to\infty} P^k(c_2) = \infty$.

Henceforce, if both critical orbits $\{P^k(c_1)\}_k$ and $\{P^k(c_2)\}_k$ are bounded, then it holds that $c_1, c_2 \in [r_{min}, r_{max}]$.

Conversely, asuume that $P(x) - x = 0$ has at least two real solutions and $c_1, c_2 \in [r_{min}, r_{max}]$. Then, $\{P^k(c_1)\}_k, \{P^k(c_2)\}_k \subset [r_{min}, r_{max}]$, since $P(x)$ is an increasing function. ■

Lemma 4. *There exists a unique real number $\alpha_1 < c_1$ such that $F'(\alpha_1) = 0$, and there exists a unique real number $\alpha_2 > c_2$ such that $F'(\alpha_2) = 0$.*

Proof.

Since $F'(x) = h(x) \cdot (h(x) + 2)$, $F''(x) = 2h'(x)(h(x) + 1)$, that is,

$$F''(x) = 2n(2n+1)^2 x^{2n-1}(x^{2n+1} + a)^{2n-1}\left(2(n+1)x^{2n+1} + a\right).$$

Set $c^* = \dfrac{-\sqrt[2n+1]{a}}{\sqrt[2n+1]{2(n+1)}}$, then $c_1 \leqq c^* \leqq c_2$. Moreover, $F''(x) < 0$ if $x < c_1$ and $F''(x) > 0$ if $x > c_2$. Since $F'(c_1) = F'(c_2) = -1$, it is easy to verify that Lemma 4 holds. ■

Lemma 5. *$F(x) = 0$ has at least two real solutions and $c_1, c_2 \in [r_{min}, r_{max}]$ if and only if $F(\alpha_1) \geqq 0$ and $F(\alpha_2) \leqq 0$.*

Proof.

Note that $F(x)$ is decreasing when $c_2 < x < \alpha_2$, increasing when $\alpha_2 < x$ and $\lim_{x\to\infty} F(x) = \infty$. Further, $F(x)$ is increasing when $x < \alpha_1$, decreasing when $\alpha_1 < x < c_1$ and $\lim_{x\to-\infty} F(x) = -\infty$. So, it is easy to verify that Lemma 5 holds. ■

Proof of Theorem 1.

Recall that $F'(x) = h(x) \cdot (h(x) + 2)$, where

$$h(x) = (2n+1)x^n(x^{2n+1} + a)^n - 1.$$

Then

$$h'(x) = n(2n+1)x^{n-1}(x^{2n+1} + a)^{n-1}(2(n+1)x^{2n+1} + a)$$

Since $h(c_1) = h(c_2) = -1$, α_1, α_2 are determined by the relations $h(\alpha_1) = h(\alpha_2) = 0$ and $\alpha_1 < c_1 \leqq c_2 < \alpha_2$.

Therefore, for any a, there is a unique $t < c_1$ such that

$$h(t) = (2n+1)t^n(t^{2n+1} + a)^n - 1 = 0,$$

that is,

$$t(t^{2n+1} + a) = \frac{1}{\sqrt[n]{2n+1}}. \tag{1}$$

For the value t,

$$F(t) = (t^{2n+1} + a)^{2n+1} + b - t = \left(\frac{1}{t\sqrt[n]{2n+1}}\right)^{2n+1} + b - t \geqq 0$$

if and only if $F(\alpha_1) \geqq 0$.

The relation (1) determines a smooth curve

$$a = \frac{1}{\kappa t} - t^{2n+1} \quad (t < 0),$$

where $\kappa = \sqrt[n]{2n+1}$. Hence, $F(\alpha_1) = 0$ if and only if

$$b = t - \frac{1}{(\kappa t)^{2n+1}}.$$

From these two relations with respect to a and b, we determine the boundary curve ℓ_1 of $C_{2n+1,\mathbb{R}}$. Let

$$\xi = a - b = \frac{1}{\kappa t} - t - t^{2n+1} + \frac{1}{(\kappa t)^{2n+1}} \quad (t<0), \tag{2}$$

$$\eta = a + b = \frac{1}{\kappa t} + t - t^{2n+1} - \frac{1}{(\kappa t)^{2n+1}} \quad (t<0), \tag{3}$$

then η is a singlevalued function $\eta(\xi)$ of ξ $(-\infty < \xi < \infty)$ and satisfies

$$\xi\left(\frac{1}{\kappa t}\right) = -\xi(t), \quad \eta\left(\frac{1}{\kappa t}\right) = \eta(t).$$

Further,

$$\frac{d\eta}{d\xi} = \frac{-\dfrac{1}{\kappa t^2} + 1 - \kappa^n t^{2n} + \dfrac{1}{\kappa^{n+1} t^{2n+2}}}{-\dfrac{1}{\kappa t^2} - 1 - \kappa^n t^{2n} - \dfrac{1}{\kappa^{n+1} t^{2n+2}}} = \frac{(\kappa t^2)^n - 1}{(\kappa t^2)^n + 1} \quad (t<0),$$

$$\frac{d^2\eta}{d\xi^2} = \frac{-4n\kappa^{2n+1} t^{4n+1}}{(\kappa^n t^{2n}+1)^3(\kappa^{n+1} t^{2n+2}+1)} \quad (t<0).$$

Hence, $\eta(\xi)$ is convex and has a unique minimal value

$$\frac{2(-\kappa^n + 1)}{\kappa^n \sqrt{\kappa}} = -\frac{4n}{(2n+1)\sqrt[2n]{2n+1}}$$

at $\xi = 0$, which corresponds to $t = -1/\sqrt{\kappa}$. Clearly, $\lim_{\xi\to\pm\infty} \eta = +\infty$. Hence, in $\xi\eta$-plane, $\eta = \eta(\xi)$ transeverses η-axis twice. By relation (3) , we know that $\eta = 0$ has only two solutions in $t < 0$, whose product is $1/\kappa$. Since $\eta < 0$ when $t = -1$, one of these solutions is less than -1 and another is between $-1/\kappa$ and 0.

The equation $\eta = 0$ of t implies

$$\kappa^{2n+1} t^{4n+2} + 1 = \kappa^{2n}(\kappa t^2 + 1).$$

Let $\kappa t^2 = -u$ then we have

$$\sum_{j=-n}^{n} u^j = (-1)^n(2n+1) \tag{4}$$

and this equation has a unique solution satisfying $u < -1$. Denote the solution by α, then $-1/\alpha$ is another solution of (4). Therefore, $\sqrt{\kappa\alpha}$, $\sqrt{\alpha/\kappa}$ are two solutions of $\eta = 0$.

Similarly, by the condition $F(\alpha_2) = 0$, we have

$$a = \frac{1}{\kappa s} - s^{2n+1} \quad (s>0),$$

and

$$b = s - \frac{1}{(\kappa s)^{2n+1}} \quad (s>0).$$

Set $t = -s$, then we get

$$-a = \frac{1}{\kappa t} - t^{2n+1} \quad (t<0),$$

and

$$-b = t - \frac{1}{(\kappa t)^{2n+1}} \quad (t < 0).$$

Hence, another boundary curve ℓ_2 of $C_{2n+1,\mathbb{R}}$ is symmetric to ℓ_1 with respect to the origin in the ab-plane. ■

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多項式族$(z^{2n+1}+a)^{2n+1}+b$の連結性集合の実断面

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要　旨

関数族 $\{(z^{2n+1}+a)^{2n+1}+b\}$ の連結性集合の実断面を表わす式を決定した.

キーワード：複素力学系, 実断面, 連結性集合, 多項式, Julia 集合