Spin-glass Theory of Random Iteration Algorithm

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(*Received September 17, 2013,*
Revised November 17, 2013)

Abstract

Cawley and Mauldin analyzed multifractal structure of a probability measure (an invariant measure) ρ induced on a Moran fractal. They introduced a system of weights as well as the probability measure and gave an example in which the multifractal structure presents spin-glass features and showed the gauge invariance.

In this paper, we consider the invariant set obtained by a random iteraition algorithm. This random algorithm introduces the assoiated pobability and weight. The multifractal decompositions of the set with respect to the pair of probability density and weight density are considered. To characterize the the Hausdorff dimension of the decomposed sets, we introduce a pair of parameters (q, s). Using these parameters we represent the formula of the Hausdorff dimension. This extension of introducing a pair of parameters gives us the freedom to investigate the spinglass phenomena of mutifractal structure. Furthermore we show the gauge invariance holds.

Keywords: Random iteration algorithms, Multifractal decompositions, Probabilities and Weights, Hausdorff dimensions, Spin-glass

1. Introduction

Cawley and Mauldin ([1]) presented a generalization of the multifractal decompositions for Moran fractals with infinite product measure. The generalization is specified by a system of nonnegative weights in the partition sum. They showed that the generalized spectrum $f(\alpha : w)$ is not concave in general.

In Section 2, we review Cawley-Maudin's results on multifractal decomposition of random iteration measures with weight and the related spinglass phenomena. In Section 3, we give some computer calculaton of the generalized spectrum of scaling indices which exhibit multi-peak curves. In Section 4, we introduce a system of weights, and prove the multifractal decomposition theory. In Section 5, we give a computer calculation of the region (α, γ) which is attainable by some (q, s). In Section 6, the gauge invariance of $f(q, s)$ is investigated.

This work was supported by JSPS KAKENHI Grant Number 23540170.

2. Multi-fractal decompositions of map specified fractals with weights (Cawley-Mauldin)

Cawley-Mauldin's formulation of multi-fractal decompositions of map specified fractals with weights is as follows:

Let J be a non-empty compact set of m-dimensinal Euclidean space $\mathbf{R}^m$ such that the closure of the interior of J is J. Assume that the diameter of J is 1, that is, $|J| = 1$. Let $T_1, \ldots, T_n$ be n contracting similarities with similarity ratios $r_1, \ldots, r_n \quad (0 < r_i < 1)$. We assume that $T_i(J) \subseteq J$ $(i = 1, \ldots, n)$ and that $T_i(J) \cap T_j(J) = \phi \quad (i \neq j)$.

The self-similar set K with respect to $\{T_1, \ldots, T_n\}$ is the non-empty compact set such that

$$K = \cup_{i=1}^{n} T_i(K).$$

Let $S^k = \{1, \ldots, n\}^k$. Then K is also expressed by

$$K = \cap_{k=0}^{\infty} \cup_{\tau \in S_k} J(\tau).$$

where $\tau = \tau(1)\tau(2)\ldots\tau(k)$ and $J(\tau) = T_{\tau(1)} \circ T_{\tau(2)} \circ \ldots T_{\tau(k)}(J)$.

The Hausdorff dimension of K is given by

$$\dim_{\mathrm{H}}(K) = d$$

where d is the unique solution of

$$\sum_{i=1}^{n} r_i^d = 1.$$

The coding space is $\Omega = \{1, \ldots, n\}^{\mathbf{N}}$, where $\mathbf{N} = \{1, 2, 3, \ldots.\}$. For each $\sigma \in \Omega$ and $k \in \mathbf{N}$, let $\sigma|k = \sigma(1)\cdots\sigma(k)$. The coding map g of Ω onto K is defined by

$$\{g(\sigma)\} = \bigcap_{k=1}^{\infty} J(\sigma|k).$$

The map g is a homeomorphism of Ω onto K.

Fix a probability vector $(p_1, \ldots, p_n)$; $\sum_{i=1}^{n} p_i = 1$, $p_i > 0 \quad i = 1, \ldots, n$, and let $\hat{\rho}$ be the corresponding infinite product measure $\prod_{k=1}^{\infty}(p_1, \ldots, p_n)$ on Ω. Let ρ be the image measure on K induced by g.

For each α $(0 \leq \alpha < \infty)$, let

$$\hat{K}_\alpha = \{\sigma \in \Omega : \lim_{k\to\infty} \log p(\sigma|k)/\log r(\sigma|k) = \alpha\}$$

and

$$K_\alpha = g(\hat{K}_\alpha),$$

where $p(\sigma|k) = \prod_{i=1}^{k} p_{\sigma(i)}$ and $r(\sigma|k) = \prod_{i=1}^{k} r_{\sigma(i)}$.

For each $q \in \mathbf{R}$, there is a unique number $\beta(q)$ such that

$$\sum_{i=1}^{n} p_i^q r_i^{\beta(q)} = 1.$$

Let

$$\alpha(q) = -\frac{d}{dq}\beta(q).$$

Assume that $(p_1, p_2, \ldots, p_n) \neq (r_1^d, r_2^d, \ldots, r_n^d)$, then it holds that $\frac{d^2}{dq^2}\beta(q) > 0$.

Cawley-Mauldin showed that
for any α $(\lambda < \alpha < \bar{\lambda})$, there exists a unique q such that

$$\alpha = -\frac{d}{dq}\beta(q)$$

where $\lambda = \min\{\log p_i / \log r_i : i = 1, \ldots, n\}$ and $\bar{\lambda} = \max\{\log p_i / \log r_i : i = 1, \ldots, n\}$.

Let

$$\hat{f}(q) = q\alpha(q) + \beta(q),$$

and

$$f(\alpha) = \hat{f}(q(\alpha)),$$

where $q(\alpha)$ is the unique q that is given by $\alpha = -\frac{d}{dq}\beta(q)$.

Cawley-Mauldin proved the following theorem.

Theorem 1 (Cawley-Mauldin [1]). *If* $(p_1, p_2, \ldots, p_n) \neq (r_1^d, r_2^d, \ldots, r_n^d)$,

$$\dim_{\mathrm{H}}(K_\alpha) = f(\alpha),$$

for each α $(\lambda < \alpha < \bar{\lambda})$.

Furtheremore it holds that

$$x \in K_\alpha \quad \text{if and only if} \lim_{\epsilon\to 0} \frac{\log \rho(B(x,\epsilon))}{\log \epsilon} = \alpha,$$

where $B(x,\epsilon)$ *is the closed ball of radius* $\epsilon > 0$ *centred at* x*,that is, if* $g(\sigma) = x$,

$$\lim_{k\to\infty} \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha \quad \text{if and only if} \quad \lim_{\epsilon\to 0} \frac{\log \rho(B(x,\epsilon))}{\log \epsilon} = \alpha. \qquad (*)$$

In the case of that $(p_1, p_2, \ldots, p_n) = (r_1^d, r_2^d, \ldots, r_n^d)$,

$$\dim_{\mathrm{H}}(K_d) = d, \quad K_\alpha = \emptyset \quad \text{for} \quad \alpha \neq d.$$

Cawley-Mauldin introduced a system of positive weights

$$w = (w_1, \ldots, w_n), \quad w_i > 0 \quad i = 1, \ldots, n.$$

For each $q \in \mathbf{R}$, there is a unique number $\beta(q : w)$ such that

$$\sum_{i=1}^{n} p_i^q w_i r_i^{\beta(q:w)} = 1.$$

Let

$$\hat{K}^{q:w} = \{\sigma \in \Omega : \lim_{k\to\infty} \log p(\sigma|k)/\log r(\sigma|k) = \alpha(q:w)$$
$$\text{and} \quad \lim_{k\to\infty} \log w(\sigma|k)/\log r(\sigma|k) = \gamma(q:w)\}$$

where

$$\alpha(q:w) = -\frac{d\beta}{dq}(q:w) = \frac{\sum_{i=1}^{n}(\log p_i)p_i^q w_i r_i^{\beta(q:w)}}{\sum_{i=1}^{n}(\log r_i)p_i^q w_i r_i^{\beta(q:w)}},$$

and

$$\gamma(q:w) = \frac{\sum_{i=1}^{n}(\log w_i)p_i^q w_i r_i^{\beta(q:w)}}{\sum_{i=1}^{n}(\log r_i)p_i^q w_i r_i^{\beta(q:w)}}$$

and $w(\sigma|k) = \prod_{i=1}^{k} w_{\sigma(i)}$.

Put

$$K^{q:w} = g(\hat{K}^{q:w}).$$

They investigated the Hausdorff dimension of $K^{q:w}$.

Let

$$\tilde{f}(q:w) = q\,\alpha(q:w) + \gamma(q:w) + \beta(q:w).$$

Theorem 2 (Cawley-Mauldin [1]). *For each $q \in \mathbf{R}$,*

$$\dim_{\mathrm{H}}(K^{q:w}) = \tilde{f}(q,w).$$

3. Some computer calculations of $f(\alpha)$ curves with weights.

Assume that $(p_1, p_2, \ldots, p_n) \neq (r_1^d, r_2^d, \ldots, r_n^d)$.

Cawley-Mauldin ([1]) showed that $\frac{d^2\beta(q,\, s)}{dq^2} > 0$. Owing to the monotone behaviours of $\alpha(q:w) = -\frac{d\beta(q,\, s)}{dq}$, for a given α ($\lambda < \alpha < \bar{\lambda}$) we have a unique $q = q(\alpha)$ such that

$$\alpha(q(\alpha):w) = \alpha.$$

Let

$$f(\alpha:w) = \tilde{f}(q(\alpha), w).$$

Cawley-Mauldin showed a computer calculation of the $f(\alpha:w)$ curve, and the model has the property that the resulting multifractal curves $f(\alpha:w)$ are no longer necessarily concave down. They state that the inclusion of an independent set of weights $\{w_i\}$ provides an additional feature of a spin-glass phenomena.

We give some computer calculation of the graphs of $f(\alpha:w)$. The dashed curves have all the weights equal to unity.

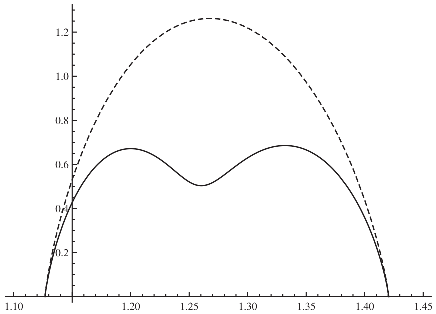

Fig. 1

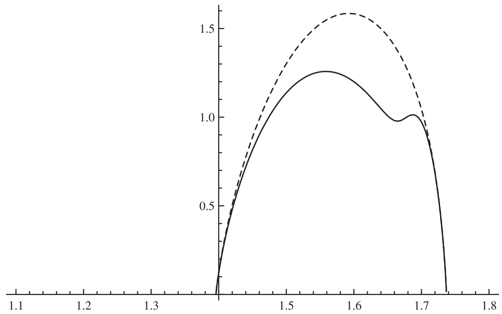

Fig. 2

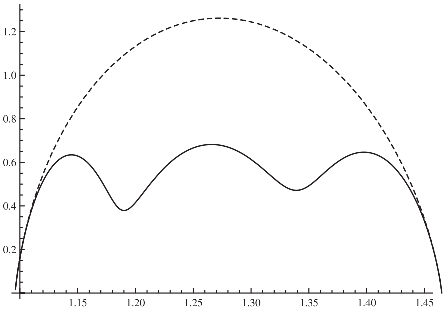

Fig. 3

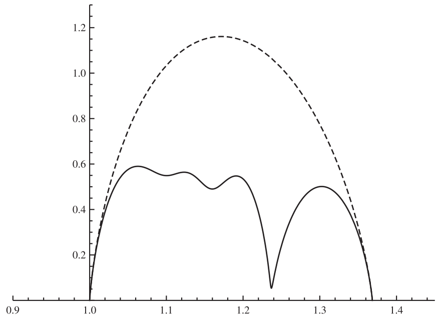

Fig. 4

Figure 1 is an example of Cawley-Mauldin in which $n = 4$ and $f(\alpha : w)$ curve has two peaks. It has $t_1 = t_2 = t_3 = t_4 = \frac{1}{3}$, $p_1 = 0.21$, $p_2 = 0.25$, $p_3 = 0.25$, $p_4 = 0.29$, and $w_1 = 0.495$, $w_2 = 0.495$, $w_3 = 0.005$, $w_4 = 0.005$.

Figure 2 is a case in which $n = 3$ and $f(\alpha : w)$ curve has two peaks. It has $t_1 = t_2 = t_3 = \frac{1}{2}$, $p_1 = 0.30$, $p_2 = 0.32$, $p_3 = 0.38$, and $w_1 = 0.453$, $w_2 = 0.545$, $w_3 = 0.002$.

Figure 3 is a case in which $n = 4$ and $f(\alpha : w)$ curve has three peaks. It has $t_1 = t_2 = t_3 = t_4 = \frac{1}{3}$, $p_1 = 0.23$, $p_2 = 0.27$, $p_3 = 0.20$, $p_4 = 0.30$, and $w_1 = 0.495$, $w_2 = 0.495$, $w_3 = 0.005$, $w_4 = 0.005$.

Figure 4 is a case in which $n = 5$ and $f(\alpha : w)$ curve has four peaks. It has $t_1 = t_2 = t_3 = t_4 = t_5 = \frac{1}{4}$, $p_1 = 0.20$, $p_2 = 0.25$, $p_3 = 0.15$, $p_4 = 0.18$, $p_5 = 0.22$ and $w_1 = 0.1433$, $w_2 = 0.00000001$, $w_3 = 0.00001999$, $w_4 = 0.856$, $w_5 = 0.00068$.

We have examples such that a graph of $f(\alpha : w)$ has $n - 1$ peaks where n is the number of the contraction.

4. Generalization of Cawley-Mauldin's formulation

In this section we generalize Cawley-Mauldin's formulation introducing another parameter s corresponding to $\{w_i\}$ so as the parameter q corresponds to $\{p_i\}$.

As before fix a probability vector $(p_1, \ldots, p_n)$; $\sum_{i=1}^{n} p_i = 1,\ p_i > 0\ \ i = 1, \ldots, n$, and let $\hat{\rho}$ be the corresponding infinite product measure $\prod_{k=1}^{\infty}(p_1, \ldots, p_n)$ on Ω. Let ρ be the image measure of $\hat{\rho}$ on K induced by g.

Now fix a weight vector $(w_1, \ldots, w_n)$; $\sum_{i=1}^{n} w_i = 1,\ w_i > 0 \quad i = 1, \ldots, n$, and let $\hat{\varrho}$ be the corresponding infinite product measure $\prod_{k=1}^{\infty}(w_1, \ldots, w_n)$ on Ω. Let ϱ be the image measure on K of $\hat{\varrho}$ induced by g. Note that Cawley and Mauldin do not assume $\sum_{i=1}^{n} w_i = 1$.

For each $(q, s) \in \mathbf{R}^2$, there is a unique number $\beta(q, s)$ such that

$$\sum_{i=1}^{n} p_i^q w_i^s r_i^{\beta(q,s)} = 1.$$

Note that

$$\beta(q, 0) = \beta(q) \quad \text{and} \quad \beta(q, 1) = \beta(q : w),$$

because

$$\sum_{i=1}^{n} p_i^q r_i^{\beta(q)} = 1 \quad \text{and} \quad \sum_{i=1}^{n} p_i^q w_i r_i^{\beta(q:w)} = 1.$$

Put

$$\alpha(q, s) = -\frac{\partial \beta}{\partial q}(q, s) \quad \text{and} \quad \gamma(q, s) = -\frac{\partial \beta}{\partial s}(q, s).$$

Let

$$\hat{K}^{(q,s)} = \{\sigma \in \Omega : \lim_{k\to\infty} \log p(\sigma|k)/\log r(\sigma|k) = \alpha(q, s)$$
$$\text{and} \quad \lim_{k\to\infty} \log w(\sigma|k)/\log r(\sigma|k) = \gamma(q, s)\}$$

and let

$$K^{(q,s)} = g(\hat{K}^{(q,s)}).$$

Put

$$\hat{f}(q, s) = q\alpha(q, s) + s\gamma(q, s) + \beta(q, s).$$

Theorem 3. *Let (α, γ) be given and suppose that there exists a pair of reals $(q, s) = (q(\alpha, \gamma), s(\alpha, \gamma))$ such that*

$$\alpha = \alpha(q, s) \quad \textit{and} \quad \gamma = \gamma(q, s).$$

Let

$$K_{(\alpha,\gamma)} = \{x \in K : \lim_{\epsilon\to 0} \frac{\log \rho(B(x, \epsilon))}{\log \epsilon} = \alpha \quad \textit{and} \quad \lim_{\epsilon\to 0} \frac{\log \varrho(B(x, \epsilon))}{\log \epsilon} = \gamma\}.$$

Then it holds that

$$K_{(\alpha,\gamma)} = K^{(q,s)}.$$

Moreover it follows that

$$\dim_{\mathrm{H}} K_{(\alpha,\gamma)} = f(\alpha,\gamma),$$

where

$$f(\alpha,\gamma) = \hat{f}(q(\alpha,\gamma), s(\alpha,\gamma)) \quad \textit{with} \quad \alpha = \alpha(q,s) \quad \textit{and} \quad \gamma = \gamma(q,s).$$

Remark. The uniqueness of a pair of reals (q, s) which attains the given (α,γ) fails in general, but the value of $f(\alpha,\gamma)$ is independent of choice of (q, s) and this fact holds from the proof of this theorem.

For the proof of this theorem we adopt the proof of Theorem 11.5 in Falconer ([2]).
First we state Proposition 2.3 in [2] for the proof of Theorem 3.

Proposition 4.1 (Falconer, Fractal Geometry [2], Proposition 2.3)**.**
Let E be a Borel set.

$$\dim_{\mathrm{H}}(E) = f$$

if there exists a finte measure ν such that $\nu(E) > 0$ and for all $x \in E$,

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = f,$$

where $B(x,r)$ is the closed ball of with centre x and radius $r > 0$.

For given $(q, s) \in \mathbf{R}^2$ and $\beta(q,\ s)$, we define a probability measure $\nu^{q,s}$ on K as follows: Let

$$\nu^{q,s}(J(\tau)) = \prod_{i=1}^{k} p^{q}_{\tau(i)} w^{s}_{\tau(i)} r^{\beta(q,\ s)}_{\tau(i)}$$

where $J(\tau) = T_{\tau(1)} \cdot T_{\tau(2)} \cdots T_{\tau(k)}(J)$ for $\tau \in S^k$, and extend to a measure on K.

Lemma 4.1. *Let (α,γ) be given and suppose that there exists a pair of reals $(q,\ s)$ $= (q(\alpha,\gamma),\ s(\alpha,\gamma))$ such that*

$$\alpha = \alpha(q,s) \quad \textit{and} \quad \gamma = \gamma(q,s).$$

Then it follows that
(a) $\nu^{q,s}(K_{(\alpha,\gamma)}) = 1$ *where $\alpha = \alpha(q,s)$ and $\gamma = \gamma(q,s)$,*
(b) *For all $x \in K_{(\alpha,\gamma)}$,*

$$\lim_{r\to 0} \frac{\log \nu^{q,s}(B(x,r))}{\log r} = \hat{f}(q,s).$$

The proof is a modification of the proof of Proposition 11.4 in Falconer ([2]). So it is proved in Appendix.

Proof of Theorem 3
The fact that $K_{(\alpha,\gamma)} = K^{(q,s)}$ is proved as the proof of the statement $(*)$ of Theorem 1.

By Lemma 4.1, we have a measure $\nu^{q,s}$ which satisfies the condition of Proposition 4.1, so the conclusion follows.

5. Attainable region of (α, γ)

It is known that the α takes values in

$$\left(\min_{1\le i\le n} \frac{\log p_i}{\log r_i}, \quad \max_{1\le i\le n} \frac{\log p_i}{\log r_i}\right)$$

and γ takes values in

$$\left(\min_{1\le i\le n} \frac{\log w_i}{\log r_i}, \quad \max_{1\le i\le n} \frac{\log w_i}{\log r_i}\right).$$

See [1].

How about the pair (α, γ)?

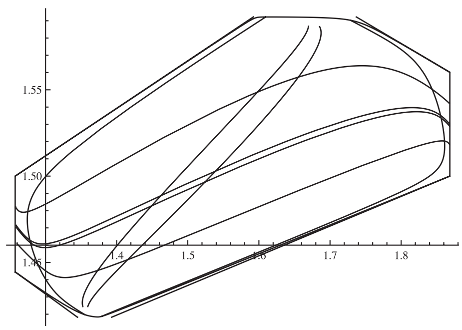

Fig. 5

We do not have the charaterization of the region of the pair (α, γ) which is attainable by some pair (q, s), that is $\alpha = \alpha(q, s)$ and $\gamma = \gamma(q, s)$.

We give an example of the trajectories of $(\alpha(q, s), \gamma(q, s))$ for fixed s's and $-100 < q < 100$, and for fixed q's and $-100 < s < 100$.

Our ratios, probabilities and weights are given by

$$n = 8, \quad t = \{1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4\},$$
$$p = \{0.11, 0.09, .075, 0.075, 0.155, 0.145, 0.175, 0.175\}$$
$$\text{and} \quad w = \{0.11, 0.11, 0.125, 0.115, 0.14, 0.14, 0.135, 0.125\}.$$

6. Gauge invariance

Cawley and Mauldin ([1]) noted the gauge invarinace of $f(\alpha : w)$. They introduce a transformation of the weights;

$$T(a,b) : w_i \to \tilde{w}_i = w_i p_i^a r_i^b, \quad i = 1, \dots, n; \quad a, b \in \mathbf{R}.$$

Recall that

$$\tilde{f}(q : w) = q\,\alpha(q : w) + \gamma(q : w) + \beta(q : w).$$

For α ($\lambda < \alpha < \bar{\lambda}$), let

$$f(\alpha : w) = \tilde{f}(q(\alpha : w) : w)$$

where $q = q(\alpha : w)$ satisfies $-\frac{d}{dq}\beta(q : w) = \alpha$.

Let

$$f(\alpha : \tilde{w}) = \tilde{f}(q(\alpha : \tilde{w}) : \tilde{w}),$$

where $q = q(\alpha : \tilde{w})$ satisfies $-\frac{d}{dq}\beta(q : \tilde{w}) = \alpha$.

They showed that

$$f(\alpha : \tilde{w}) = f(\alpha : w),$$

and call it gauge invariant property.

In their formulation, they do not assume the condition $\sum_{i=1}^n w_i = 1$.

We show the gauge invariance in our formulation. In our setting we assume that $\sum_{i=1}^n w_i = 1$, $w_i > 0$ ($i = 1, \dots, n$), and so the transformation $T(a,b)$ satisfies that $\sum_{i=1}^n \tilde{w}_i = \sum_{i=1}^n w_i p_i^a r_i^b = 1$ and so

$$b = \beta(a, 1).$$

We consider a transformation of the weights;

$$\bar{T}(a) : w_i \to \bar{w}_i = w_i p_i^a r_i^{\beta(a,1)}, \quad i = 1, \dots, n; \quad a \in \mathbf{R}.$$

Recall that

$$\hat{f}(q, s) = q\alpha(q, s) + s\gamma(q, s) + \beta(q, s).$$

Under these weights $\{\bar{w}_i, i = 1, \dots, n\}$, the corresponding $\beta(q, s : \bar{w})$, $\alpha(q, s : \bar{w})$, $\gamma(q, s : \bar{w})$ and $\hat{f}(q, s : \bar{w})$ are related to $\beta(q, s : w)$, $\alpha(q, s : w)$, $\gamma(q, s : w)$ and $\hat{f}(q, s : w)$ as follows:

$$\begin{aligned}
\beta(q, s : \bar{w}) &= \beta(q + as, s : w) - \beta(a, 1)s \\
\alpha(q, s : \bar{w}) &= \alpha(q + as, s : w) \\
\gamma(q, s : \bar{w}) &= \alpha(q + as, s : w)a + \gamma(q + as, s : w) + \beta(a, 1).
\end{aligned}$$

It holds that

$$\alpha(q - as, s : \bar{w}) = \alpha(q, s : w).$$

We have that

$$\hat{f}(q - as, s : \bar{w}) = \hat{f}(q, s : w),$$

because $\hat{f}(q - as, s : \bar{w}) = \alpha(q - as, s : \bar{w})(q - as) + \gamma(q - as, s : \bar{w})s + \beta(q - as, s : \bar{w}) = \alpha(q, s : w)(q - as) + (\alpha(q, s : w)a + \gamma(q, s : w) + \beta(a, 1))s + \beta(q, s : w) - \beta(a, 1)s = \alpha(q, s : w)q + \gamma(q, s : w)s + \beta(q, s : w) = \hat{f}(q, s : w)$.

For α ($\lambda < \alpha < \bar{\lambda}$), let

$$f(\alpha, s : w) = \hat{f}(q(\alpha, s : w), s : w)$$

where $q = q(\alpha, s : w)$ satisfies $-\frac{\partial}{\partial q}\beta(q, s : w) = \alpha$, and let

$$f(\alpha, s : \bar{w}) = \hat{f}(q(\alpha, s : \bar{w}), s : \bar{w}),$$

where $q = q(\alpha, s : \bar{w})$ satisfies $-\frac{\partial}{\partial q}\beta(q, s : \bar{w}) = \alpha$. Note that $q(\alpha, s : \bar{w}) = q(\alpha, s : w) - as$.

Then we have the following gauge invariance;

$$f(\alpha, s : \bar{w}) = f(\alpha, s : w),$$

because $f(\alpha, s : \bar{w}) = \hat{f}(q(\alpha, s : \bar{w}), s : \bar{w}) = \hat{f}(q(\alpha, s : w) - as, s : \bar{w}) = \hat{f}(q(\alpha, s : w), s : w) = f(\alpha, s : w)$.

Appendix

For the proof of Lemma 4.1, we state another lemma.

Let

$$\Phi(q, s, \beta) = \sum_{i=1}^{n} p_i^q w_i^s r_i^\beta$$

for real numbers q, s and β. By the definition of $\beta(q, s)$, we have $\Phi(q, s, \beta(q, s)) = 1$.

Lemma A.1. *Let $\epsilon > 0$. It holds that for $\alpha = -\frac{\partial\beta}{\partial q}(q, s)$,*

$$\Phi(q + \delta, s, \beta(q, s) + (-\alpha + \epsilon)\delta) < 1,$$
$$\Phi(q - \delta, s, \beta(q, s) + (\alpha + \epsilon)\delta) < 1,$$

and for $\gamma = -\frac{\partial\beta}{\partial s}(q, s)$,

$$\Phi(q, s + \delta, \beta(q, s) + (-\gamma + \epsilon)\delta) < 1,$$
$$\Phi(q, s - \delta, \beta(q, s) + (\gamma + \epsilon)\delta) < 1$$

for a small $\delta > 0$.

Proof of Lemma A.1

It holds that $\Phi(q + \delta, s, \beta(q + \delta, s)) = 1$. Note that $\beta(q + \delta, s) < \beta(q, s) + (-\alpha + \epsilon)\,\delta$ if $\delta (> 0)$ is small, because $\alpha = -\frac{\partial\beta}{\partial q}(q, s)$. Φ is decreasing in its third argument, so we have

$$\Phi(q + \delta, s, \beta(q, s) + (-\alpha + \epsilon)\delta) < 1.$$

Other inequalities are proved similarly.

Proof of Lemma 4.1

(a)

For $x \in K$, we write $J^k(x)$ for the set $J(\tau)$ ($\tau \in S^k$) which contains x.

Let $0 < \epsilon < \alpha$. Then for $\delta > 0$, we have

$$\begin{aligned}
\nu^{q,s}(x \in K : \rho(J^k(x)) \geq |J^k(x)|^{\alpha-\epsilon}) &= \nu^{q,s}(x \in K : 1 \leq \rho(J^k(x))^{\delta} |J^k(x)|^{(\epsilon-\alpha)\delta}) \\
&\leq \int \rho(J^k(x))^{\delta} |J^k(x)|^{(\epsilon-\alpha)\delta} d\nu^{q,s}(x) \\
&= \sum_{\tau \in S^k} \rho(J^k(\tau))^{\delta} |J^k(\tau)|^{(\epsilon-\alpha)\delta} \nu^{q,s}(\tau) \\
&= \sum_{\tau \in S^k} \prod_{i=1}^{k} (p_{\tau(i)})^{\delta} (r_{\tau(i)})^{(\epsilon-\alpha)\delta} (p_{\tau(i)})^{q} (w_{\tau(i)})^{s} (r_{\tau(i)})^{\beta(q,s)} \\
&= \left(\sum_{i=1}^{n} (p_{\tau(i)})^{q+\delta} (w_{\tau(i)})^{s} (r_{\tau(i)})^{\beta(q,s)+(\epsilon-\alpha)\delta)} \right)^{k} \\
&= (\Phi(q+\delta, s, \beta(q,s) + (\epsilon-\alpha)\delta))^{k}.
\end{aligned}$$

By Lemma A.1, it holds that

$$\nu^{q,s}(x \in K : \rho(J^k(x)) \geq |J^k(x)|^{\alpha-\epsilon}) < \eta^k,$$

where $0 < \eta < 1$, so we have

$$\nu^{q,s}(x \in K : \rho(J^k(x)) \geq |J^k(x)|^{\alpha-\epsilon} \quad \text{for some} \quad k \geq k_0) \leq \sum_{k_0}^{\infty} \eta^k < \infty.$$

It follows that for $\nu^{q,s}$ – a.e. x,

$$\liminf_{k\to\infty} \frac{\log \rho(J^k(x))}{\log |J^k(x)|} \geq \alpha - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, for $\nu^{q,s}$ – a.e. x,

$$\liminf_{k\to\infty} \frac{\log \rho(J^k(x))}{\log |J^k(x)|} \geq \alpha.$$

Similarily we obtain that

$$\limsup_{k\to\infty} \frac{\log \rho(J^k(x))}{\log |J^k(x)|} \leq \alpha,$$

by using

$$\Phi(q-\delta, s, \beta(q,s) + (\alpha+\epsilon)\,\delta) < 1$$

in Lemma A.1.

It means that for $\nu^{q,s}$ – a.e. $x \in K$,

$$\lim_{k\to\infty} \frac{\log \rho(J^k(x))}{\log |J^k(x)|} = \alpha.$$

Furthermore we obtain for $\nu^{q,s}$ − a.e. $x \in K$,

$$\lim_{k\to\infty} \frac{\log \varrho(J^k(x))}{\log |J^k(x)|} = \gamma,$$

by Lemma A.1.

By the same argument as the statement (∗) of Theorem 1, we have

$$\lim_{r\to 0} \frac{\log \rho(B(x,r))}{\log r} = \lim_{k\to\infty} \frac{\log \rho(J^k(x))}{\log |J^k(x)|} \quad \text{and} \quad \lim_{r\to 0} \frac{\log \varrho(B(x,r))}{\log r} = \lim_{k\to\infty} \frac{\log \varrho(J^k(x))}{\log |J^k(x)|}$$

and so

$$\nu^{q,s}(K_{(\alpha,\gamma)}) = 1.$$

(b)

Note that

$$\frac{\log \nu^{q,s}(J^k(x))}{\log |J^k(x)|} = q\frac{\log \rho(J^k(x))}{\log |J^k(x)|} + s\frac{\log \varrho(J^k(x))}{\log |J^k(x)|} + \beta(q,s)\frac{\log |J^k(x)|}{\log |J^k(x)|}.$$

For all $x \in K_{(\alpha,\gamma)}$,

$$\frac{\log \nu^{q,s}(J^k(x))}{\log |J^k(x)|} \to q\alpha + s\gamma + \beta$$

as $k \to \infty$.

By the same argument as the statement (∗) of Theorem 1, we have

$$\lim_{r\to 0} \frac{\log \nu^{q,s}(B(x,r))}{\log r} = \lim_{k\to\infty} \frac{\log \nu^{q,s}(J^k(x))}{\log |J^k(x)|}.$$

So it holds that,

$$\lim_{r\to 0} \frac{\log \nu^{q,s}(B(x,r))}{\log r} = q\alpha + s\gamma + \beta = \hat{f}(q,s).$$

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ランダム反復アルゴリズムのスピングラス理論

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要　旨

Cawley-Mauldin は Moran フラクタル上の確率測度 ρ のマルチフラクタル構造を解析している．彼らは確率測度に加えて重み測度を導入し，そのマルチフラクタル構造がスピングラスの性質を持つ例を与え，ゲージ不変性を示している．

この論文ではランダム反復アルゴリズムによって得られる集合を考える．このランダム反復アルゴリズムに対応する確率と重みを考え，確率と重みのペアーに対応するマルチフラクタル分解を考察する．分解要素のハウスドルフ次元を特徴づけるために，パラメータの対 (q, s) を導入する．このパラメータの対を使ってハウスドルフ次元の公式を表す．この拡張はマルチフラクタル構造のスピングラス現象を解析するための自由性を与える．更にゲージ不変性が成立することを示す．

キーワード：ランダム反復アルゴリズム，マルチフラクタル分解，確率と重み，スピングラス，ゲージ不変性