Multifractal decomposition of mutual-recursive sets corresponding to a probability and a weight

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Abstract

A random iteration algorithm for mutual-recursive sets (graph-directed sets) is considered. We deal with a pair of a probability and a weight on the mutual-recursive sets. The multifractal decompositions with respect to the pair of probability density and weight density are investigated. Therefore, we introduce a pair of parameters $(q,\ s)$. Using these parameters, the Hausdorff dimension and the Packing dimension of the set in the mutual-recursive set, of which the densities of probability and weight are specified, are charactrized. This extension of introducing a pair of parameters gives us the freedom to invetigate the spinglass phenomena of multifractal structure as indicated in [4]. This work is an extension of [1], [3] and [4].

Keywords: Random iteration algorithms, Mutual-recursive sets, Multifractal decompositions, Hausdorff dimensions, Probabilities and Weights

1. The setting

We follow the formulation of Edgar-Mauldin ([1]).

Let (V, E) be a directed multigraph; a $v \in V$ is a vertex of the graph; an element $e \in E$ is an edge of the graph. For $u, v \in V$, a subset E_{uv} is the set of the edges from u to v and let $E_u := \cup_{v \in V} E_{uv}$.

A path in the graph is a finite string $\gamma = e_1 e_2 \cdots e_k$ of edges such that the terminal vertex of each edge e_i is the initial vertex of the next edge e_{i+1}.

$E_{uv}^{(k)} := \{\gamma = e_1 e_2 \cdots e_k :$ a path that begins at u and ends at $v\}$
$E_u^{(k)} := \bigcup_{v \in V} E_{uv}^{(k)}$
$E_u^{(*)} := \bigcup_{k=0}^{\infty} E_u^{(k)}$
$E^{(*)} := \bigcup_{u \in V} E_u^{(*)}$.

A path that begins and ends at the same vertex is called a cycle. A cycle with no repeated vertex is a simple cycle.

We assume that the graph (V, E) is **strongly conneted**, that is, there exists a path from any vertex to any other vertex. Furthermore we assume that there are at least two edges leaving each vertex, that is, $\#E_u \geqq 2$ for any $u \in V$. This assumption assures later that $0 < p(e) < 1$.

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Let J_u ($u \in V$) be non-empty compact subsets of $\mathbf{R}^n$ such that the closure of the interior of J_u is J_u and the diameter of J_u is 1 for any $u \in V$ for convenience.

A similarity θ_e is specified for each $e \in E$ with ratio $r(e)$ with $0 < r(e) < 1$.

Assume that $\theta_e(J_v) \subset J_u$ if $e \in E_{uv}$, and $\theta_e(J_v) \cap \theta_{\hat{e}}(J_{\hat{v}}) = \emptyset$ for any $e \in E_{uv}$, $\hat{e} \in E_{u\hat{v}} (e \neq \hat{e})$ for any u, v and $\hat{v}$ in V.

If $\gamma = e_1 \cdots e_k \in E_{uv}^{(k)}$ is a path, let $J(\gamma) = \theta_{e_1}\theta_{e_2} \ldots \theta_{e_k}(J_v) \subset J_u$ and $r(\gamma) = r(e_1) \cdots r(e_k)$.

Mutual-recursive sets (or, graph-directed sets) are

$$K_u = \bigcap_{k=0}^{\infty} \bigcup_{\gamma \in E_u^{(k)}} J(\gamma), \quad u \in V.$$

Let $A(s)$ be a square matrix such that the entry in row u and column v is

$$A_{uv}(s) = \sum_{e \in E_{uv}} r(e)^s.$$

The Hausdorff dimension of all the sets K_u is the unique non-negative number of d such that the matrix $A(d)$ has spectral radius 1.

The models

Write $E_u^{(\omega)}$ for the set of all infinite strings with alphabet E where the initial vertex of the first edge is u and the terminal vertex of each edge is the initial vertex of the next edge. For each $\gamma \in E^{(*)}$, the cylinder $[\gamma]$ is the set of all infinite strings $\sigma \in E_u^{(\omega)}$ that begin with γ.

There is a model map $h_u : E_u^{(\omega)} \to \mathbf{R}^n$ defined so that

$$h_u(\sigma) = \bigcap_{k=1}^{\infty} J(\sigma|k),$$

where $\sigma|k = e_1 \ldots e_k$ for $\sigma = e_1 \ldots e_k \ldots$ and $e_1 \in E_u$.

Probability and weight

In this paper we consider probability $p(e)$ and weight $w(e)$ such that

$$\sum_{v \in V} \sum_{e \in E_{uv}} p(e) = 1, \quad \sum_{v \in V} \sum_{e \in E_{uv}} w(e) = 1.$$

Note that $0 < p(e) < 1,\ 0 < w(e) < 1$, because $\#E_u \geq 2$.

We define probabilities and weights of paths; If $\gamma = e_1 e_2 \ldots e_k$, then

$$p(\gamma) = p(e_1) \cdots p(e_k), \quad w(\gamma) = w(e_1) \cdots w(e_k).$$

There is a unique measure $\hat{\rho}_u$ on $E_u^{(\omega)}$ with $\hat{\rho}_u([\gamma]) = p(\gamma)$ for all $\gamma \in E_u^{(*)}$.

A measure ρ_u on K_u ($u \in V$) is defined by

$$\rho_u(F) = \hat{\rho}_u(h_u^{-1}(F))$$

for a measurable set $F \subset \mathbf{R}^n$.

Similarly we define a weight $\hat{\varrho}_u$ on $E_u^{(\omega)}$ and ϱ_u on K_u ($u \in V$); $\hat{\varrho}_u$ on $E_u^{(\omega)}$ with $\hat{\varrho}_u([\gamma]) = w(\gamma)$ for all $\gamma \in E_u^{(*)}$, and $\varrho_u(F) = \hat{\varrho}_u(h_u^{-1}(F))$ for a measurable set $F \subset \mathbf{R}^n$.

Hausdorff dimension

Let $F \subseteq \mathbf{R}^n$ be a set. For fixed positive numbers s and δ, let

$$H^s_\delta(F) = \inf \sum_i (\operatorname{diam} A_i)^s,$$

where the infimum is over all countable families $\{A_i\}_{i=1}^{\infty}$ of sets with $\cup_i A_i \supseteq F$ and $\operatorname{diam} A_i < \delta$ for all i. Define the $s-$dimensional outer Hausdorff measure of F by

$$H^s(F) = \lim_{\delta \downarrow 0} H^s_\delta(F) = \sup_{\delta > 0} H^s_\delta(F).$$

There is a unique value d, such that

$$H^s(F) = \begin{cases} \infty & \text{if} \quad s < d \\ 0 & \text{if} \quad s > d \end{cases}$$

This critical value d is the Hausdorff dimension of the set F. We write $d = \dim_{\mathrm{H}} F$.

Packing dimension

Let $F \subseteq \mathbf{R}^n$ be a set.

For fixed positive numbers s and ϵ, let

$$\tilde{P}^s_\epsilon(F) = \sup \sum_i (2\epsilon_i)^s,$$

where the supremum is over all countable disjoint families $\{B_{\epsilon_i}(x_i)\}_{i=1}^{\infty}$ of balls with $\epsilon_i < \epsilon$ and $x_i \in F$ where $B_\epsilon(x) = \{y \in \mathbf{R}^n : |y - x| \leq \epsilon\}$. Define the $s-$dimensional packing pre-measure of F by

$$\tilde{P}^s(F) = \lim_{\epsilon \downarrow 0} \tilde{P}^s_\epsilon(F) = \inf_{\epsilon > 0} \tilde{P}^s_\epsilon(F).$$

Define the s-dimensional packing measure of F by

$$P^s(F) = \inf \sum_i^{\infty} \tilde{P}^s_\epsilon(F_i),$$

where the infimum is over all countable families $\{F_i\}_{i=1}^{\infty}$ of sets with $\cup_i F_i \supseteq F$.

There is a unique value d, such that

$$P^s(F) = \begin{cases} \infty & \text{if} \quad s < d \\ 0 & \text{if} \quad s > d \end{cases}$$

This critical value d us the packing dimension of the set F. We write $d = \dim_{\mathrm{P}} F$.

Multifractal decomposition

We consider multifractal decomposition of the pair (ρ_u, ϱ_u) of the measure and the weight on K_u where $u \in V$

Given two reals α_p and α_w, we set

$$\hat{K}_u^{(\alpha_p,*)} = \{\sigma \in E_u^{(\omega)} : \lim_{k\to\infty} \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha_p\},$$
$$\hat{K}_u^{(*,\alpha_w)} = \{\sigma \in E_u^{(\omega)} : \lim_{k\to\infty} \frac{\log w(\sigma|k)}{\log r(\sigma|k)} = \alpha_w\},$$
$$\hat{K}_u^{(\alpha_p,\alpha_w)} = \{\sigma \in E_u^{(\omega)} : \lim_{k\to\infty} \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha_p \quad \text{and} \quad \lim_{k\to\infty} \frac{\log w(\sigma|k)}{\log r(\sigma|k)} = \alpha_w\}.$$

Let

$$K_u^{(\alpha_p,*)} = h_u(\hat{K}_u^{(\alpha_p,*)}),$$
$$K_u^{(*,\alpha_w)} = h_u(\hat{K}_u^{(*,\alpha_w)}),$$
$$K_u^{(\alpha_p,\alpha_w)} = h_u(\hat{K}_u^{(\alpha_p,\alpha_w)}).$$

$\{K_u^{(\alpha_p,*)}\}_{\alpha_p}$, $\{K_u^{(*,\alpha_w)}\}_{\alpha_w}$, and $\{K_u^{(\alpha_p,\alpha_w)}\}_{(\alpha_p,\alpha_w)}$ are called the multi-fractal decomposition with respect to ρ, ϱ, and (ρ, ϱ) respectively.

The assumption that $\theta_e(J_v) \bigcap \theta_{\hat{e}}(J_{\hat{v}}) = \emptyset$ for any $e, \hat{e} \in E_u$ with the exception of the case when $e = \hat{e}$ implies that

$$K_u^{(\alpha_p,*)} = \{x \in K_u : \lim_{r\to 0} \frac{\log \rho_u(B_r(x))}{\log |B_r(x)|} = \alpha_p\},$$
$$K_u^{(*,\alpha_w)} = \{x \in K_u : \lim_{r\to 0} \frac{\log \varrho_u(B_r(x))}{\log |B_r(x)|} = \alpha_w\},$$
$$K_u^{(\alpha_p,\alpha_w)} = \{x \in K_u : \lim_{r\to 0} \frac{\log \rho_u(B_r(x))}{\log |B_r(x)|} = \alpha_p \quad \text{and} \quad \lim_{r\to 0} \frac{\log \varrho_u(B_r(x))}{\log |B_r(x)|} = \alpha_w\}$$

where $B_\epsilon(x)$ is the closed ball of radius ϵ centred at x and so $|B_\epsilon(x)| = 2\epsilon$. These facts are proved in Edgar-Mauldin [1] (p.610).

The Hausdorff dimensions and the Packing dimensions of the multifractal components $K_u^{(\alpha_p,*)}$, $K_u^{(*,\alpha_w)}$ and $K_u^{(\alpha_p,\alpha_w)}$ are computed as follows.

Let $A(q, s, \beta)$ be a square $V \times V$ matrix. The entry is

$$A_{uv}(q, s, \beta) = \sum_{e\in E_{uv}} p(e)^q w(e)^s r(e)^\beta \quad (u, v \in V).$$

For given (q, s), there is a unique β such that $A(q, s, \beta)$ has spectral 1. This β is an analytic function $\beta(q, s)$ of q and s.

Define

$$\alpha_p(q, s) = -\frac{\partial\beta(q, s)}{\partial q}, \quad \alpha_w(q, s) = -\frac{\partial\beta(q, s)}{\partial s},$$

and

$$f(q, s) = \alpha_p(q, s)q + \alpha_w(q, s)s + \beta(q, s).$$

Theorem 1 (Extension of Theorem 1.6 of Edgar and Mauldin [1]).

Let (V, E) be a strongly connected directed multigraph. Let $r(e)$ with $0 < r(e) < 1$ be a system of ratios for the graph, and let $p(e)$ and $w(e)$, with $0 < p(e) < 1$ and $0 < w(e) < 1$, be two systems of transition probabilities for the graph. They define $\hat{\rho}_u$ and $\hat{\varrho}_u$ on the models $E_u^{(\omega)}$ where $u \in V$. Let q, s, β, α_p, α_w be numbers above. Then for each $u \in V$, the Hausdorff dimension and the packing dimension of multifractal components $K_u^{(\alpha_p,*)}$, $K_u^{(*,\alpha_w)}$, and $K_u^{(\alpha_p,\alpha_w)}$ are given by

$$\dim_{\mathrm{H}} K_u^{(\alpha_p,*)} = \dim_{\mathrm{P}} K_u^{(\alpha_p,*)} = f(q, 0)$$

where $\alpha_p = -\frac{\partial\beta(q,0)}{\partial q}$ for some q,

$$\dim_{\mathrm{H}} K_u^{(*,\alpha_w)} = \dim_{\mathrm{P}} K_u^{(*,\alpha_w)} = f(0, s)$$

where $\alpha_w = -\frac{\partial\beta(0,s)}{\partial s}$ for some s,

$$\dim_{\mathrm{H}} K_u^{(\alpha_p,\alpha_w)} = \dim_{\mathrm{P}} K_u^{(\alpha_p,\alpha_w)} = f(q, s)$$

where $\alpha_p = -\frac{\partial\beta(q,s)}{\partial q}$ and $\alpha_w = -\frac{\partial\beta(q,s)}{\partial w}$ for some q and s.

The Hausdorff dimensions and the packing dimensions do not depend on the choice of a pair of q and s.

This theorem is proved in Section 3.

2. Auxiliary functions

Recall that $A(q, s, \beta)$ is the matrix with the entry

$$A_{uv}(q, s, \beta) = \sum_{e \in E_{uv}} p(e)^q w(e)^s r(e)^\beta \quad (u, v \in V).$$

Let $\Phi(q, s, \beta)$ be the spectral radius of $A_{uv}(q, s, \beta)$.

Proposition 2.1 (Extension of Proposition 3.1 of Edgar and Mauldin [1]).

(i) $\Phi(q, s, \beta) : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to (0, \infty)$ *is continuous.*

(ii) $\Phi(q, s, \beta)$ *is strictly decreasing in each variable seperately.*

(iii) *For fixed q,s we have* $\lim_{\beta\to\infty} \Phi(q, s, \beta) = 0$ *and* $\lim_{\beta\to-\infty} \Phi(q, s, \beta) = \infty$. *For fixed β and s we have* $\lim_{q\to\infty} \Phi(q, s, \beta) = 0$ *and* $\lim_{q\to-\infty} \Phi(q, s, \beta) = \infty$. *For fixed β and q we have* $\lim_{s\to\infty} \Phi(q, s, \beta) = 0$ *and* $\lim_{s\to-\infty} \Phi(q, s, \beta) = \infty$.

(iv) $\Phi(q, s, \beta)$ *is log-convex; if* $q_1, q_2, s_1, s_2, \beta_1, \beta_2 \in \mathbf{R}$, $a_1, a_2 \geq 0$, $a_1 + a_2 = 1$, *then*

$$\Phi(a_1q_1 + a_2q_2, a_1s_1 + a_2s_2, a_1\beta_1 + a_2\beta_2) \leq \Phi(q_1, s_1, \beta_1)^{a_1}\Phi(q_2, s_2, \beta_2)^{a_2}.$$

Proof. See the proof of Proposition 3.1 of Edgar and Mauldin [1].

Proposition 2.2 (Extension of Proposition 3.2 of Edgar and Mauldin [1]).

Let $\beta = \beta(q, s)$ be defined by $\Phi(q, s, \beta) = 1$. Then

(i) $\beta(q,s)$ *is an analytic function of the variable q, and an analytic function of the variable s.*

(ii) $\beta(q,s)$ *is strictly decreasing ; if* $q_1 < q_2$*, then* $\beta(q_1,s) > \beta(q_2,s)$*, and if* $s_1 < s_2$*, then* $\beta(q,s_1) > \beta(q,s_2)$*.*

(iii) $\lim_{q\to-\infty}\beta(q,s) = \infty$ *and* $\lim_{q\to\infty}\beta(q,s) = -\infty$. $\lim_{s\to-\infty}\beta(q,s) = \infty$ *and* $\lim_{s\to\infty}\beta(q,s) = -\infty$

(iv) $\beta(q,s)$ *is a convex funtion with respect to two variables q,s; if* $a_1, a_2 \geq 0$*,* $a_1 + a_2 = 1$*, then*

$$\beta(a_1q_1 + a_2q_2, a_1s_1 + a_2s_2) \leq a_1\beta(q_1,s_1) + a_2\beta(q_2,s_2).$$

Proof. See the proof of Proposition 3.2 of Edgar and Mauldin [1].

Recall that $p(e)$ and $w(e)$ satisfy

$$\sum_{v\in V}\sum_{e\in E_{uv}} p(e) = 1 \quad \text{and} \quad \sum_{v\in V}\sum_{e\in E_{uv}} w(e) = 1,$$

for all $u \in V$.

By the Perron-Frobenius theorem, the spectral radius of $A(1,0,0)$ is 1 and so $\beta(1,0) = 0$. And the spectral radius of $A(0,1,0)$ is 1 and so $\beta(0,1) = 0$.

Let $\{\xi_v(q,s)\}_{v\in V}$ be the unique positive eigen vector of $A(q,s,\beta(q,s))$ such that

$$\sum_{v\in V}\sum_{e\in E_{uv}} p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v(q,s) = \xi_u(q,s) \quad \text{for all} \quad u \in V, \quad \text{and} \quad \sum_{v\in V}\xi_v(q,s) = 1.$$

The entries $\{\xi_v(q,s)\}_{v\in V}$ are analytic functions of q and s.

Let $\{\lambda_u(q,s)\}_{u\in V}$ be the unique positive left eigen vector for which

$$\sum_{u\in V}\sum_{e\in E_{uv}} \lambda_u(q,s)p(e)^q w(e)^s r(e)^{\beta(q,s)} = \lambda_v(q,s) \quad \text{for all} \quad v \in V,$$

and

$$\sum_{u\in V}\xi_u(q,s)\lambda_u(q,s) = 1.$$

The entries $\{\lambda_u(q,s)\}_{u\in V}$ are analytic functions of q and s.

Let $\xi_v = \xi_v(q,s), \lambda_u = \lambda_u(q,s)$. Following the argument in Edgar and Mauldin [1], we have

$$\frac{\partial\beta}{\partial q} = -\frac{\sum_u\sum_v\sum_{e\in E_{uv}}(\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v)\log p(e)}{\sum_u\sum_v\sum_{e\in E_{uv}}(\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v)\log r(e)} < 0.$$

Let $\alpha_p = -\frac{\partial\beta}{\partial q}$, then $\alpha_p > 0$ and

$$\alpha_p = \frac{\sum_u\sum_v\sum_{e\in E_{uv}}(\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v)\log p(e)}{\sum_u\sum_v\sum_{e\in E_{uv}}(\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v)\log r(e)}.$$

Similarly we have

$$\frac{\partial\beta}{\partial s} = -\frac{\sum_u\sum_v\sum_{e\in E_{uv}}(\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v)\log w(e)}{\sum_u\sum_v\sum_{e\in E_{uv}}(\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)}\xi_v)\log r(e)} < 0.$$

Let $\alpha_w = -\frac{\partial\beta}{\partial s}$, then $\alpha_w > 0$ and

$$\alpha_w = \frac{\sum_u \sum_v \sum_{e\in E_{uv}} (\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)} \xi_v) \log w(e)}{\sum_u \sum_v \sum_{e\in E_{uv}} (\lambda_u p(e)^q w(e)^s r(e)^{\beta(q,s)} \xi_v) \log r(e)}.$$

If $\gamma = e_1 e_2 \dots e_k \in E^{(k)}$,

$$\eta_p(\gamma) = \frac{p(\gamma)}{r(\gamma)} = \frac{p(e_1)p(e_2)\dots p(e_k)}{r(e_1)r(e_2)\dots r(e_k)},$$
$$\eta_w(\gamma) = \frac{w(\gamma)}{r(\gamma)} = \frac{w(e_1)w(e_2)\dots w(e_k)}{r(e_1)r(e_2)\dots r(e_k)}.$$

Let

$$\eta_p^{min} = \min\{\eta_p(\gamma) : \gamma \quad \text{is a simlpe cycle}\},$$
$$\eta_p^{max} = \max\{\eta_p(\gamma) : \gamma \quad \text{is a simlpe cycle}\},$$
$$\eta_w^{min} = \min\{\eta_w(\gamma) : \gamma \quad \text{is a simlpe cycle}\},$$
$$\eta_w^{max} = \max\{\eta_w(\gamma) : \gamma \quad \text{is a simlpe cycle}\}.$$

By Proposition 2.2, $\beta(q,s)$ is a convex function of q and s, so $\frac{\partial^2\beta}{\partial q^2} \ge 0$ and $\frac{\partial^2\beta}{\partial s^2} \ge 0$. This means that $\frac{\partial\alpha_p}{\partial q} \le 0$ and $\frac{\partial\alpha_w}{\partial s} \le 0$.

Proposition 2.3 (Extension of Proposition 3.3 of Edgar and Mauldin [1])**.**

Let $(x_v)_{v\in V}$ be the Perron numbers; i.e., $x_v > 0$ and

$$\sum_{v\in V} \sum_{e\in E_{uv}} r(e)^d x_v^d = x_u^d \quad \textit{for all } u \in V.$$

(A) *Suppose that $p(e) = w(e) = (x_u^{-1} r(e) x_v)^d$, for all $u, v \in V$ and $e \in E_{uv}$. Then*

(i) *$\beta(q,s) = d - dq - ds$.*

(ii) *$\alpha_p(q,s) = d$ and $\alpha_w(q,s) = d$.*

(iii) *$f(q,s) = d$.*

(iv) *$K_u^{(d,d)} = K_u$ and $K_u^{(\alpha_p,\alpha_w)} = \emptyset$ for all $(\alpha_p,\alpha_w) \ne (d,d)$.*

(B) *Suppose that $p(e) = (x_u^{-1} r(e) x_v)^d$ for all $u, v \in V$ and $e \in E_{uv}$, and $w(e) \ne (x_u^{-1} r(e) x_v)^d$ for at least one edge e. Then*

(i) *$\beta(q,s) = -dq + \phi_w(s)$ where $\phi_w(s)$ is a function of s defined by $\Phi(0, s, \phi_w(s)) = 1$. $\phi_w(s)$ is a strictly convex function of s.*

(ii) *$\alpha_p(q,s) = d$ is constant. $\alpha_w(q,s)(= -\frac{d}{ds}\phi_w(s))$ does not depend on q and is a strictly decreasing funtion of s. So if we fix q we may consider s as a function of α_w defined on an interval $(\eta_w^{min}, \eta_w^{max})$.*

(iii) *$f(q,s) = s\alpha_w(s) + \phi_w(s)$ does not depend on q and is a strictly concave function of α_w.*

(iv) *$K_u^{(*,\alpha_w)} = K_u^{(d,\alpha_w)} \ne \emptyset$ if and only if $\eta_w^{min} \le \alpha_w \le \eta_w^{max}$.*

(B') *Suppose that $w(e) = (x_u^{-1} r(e) x_v)^d$, and for all $u, v \in V$ and $e \in E_{uv}$, and $p(e) \ne (x_u^{-1} r(e) x_v)^d$ for at least one edge e. Then*

(i) *$\beta(q,s) = -ds + \phi_p(q)$ where $\phi_p(q)$ is a function of q defined by $\Phi(q, 0, \phi_p(q)) = 1$. $\phi_p(q)$ is a strictly convex function of q.*

(ii) $\alpha_w = d$ *is constant.* α_1 *does not depend on* q *and is a strictly convex funtion of* q*. So we may consider* q *as a function of* α_p *defined on an interval* $(\eta_p^{min}, \eta_p^{max})$.

(iii) $f(q,s) = q\,\alpha_p + \phi_w(s)$*, so* $f(q,s)$ *does not depend on* q *and is a strictly concave function of* α_w.

(iv) $K_u^{(\alpha_p,*)} \neq \emptyset$ *if and only if* $\eta_p^{min} \leq \alpha_p \leq \eta_p^{max}$. $K_u^{(\alpha_p,\alpha_w)} \neq \emptyset$ *if* $\eta_p^{min} \leq \alpha_p \leq \eta_p^{max}$.

(C) *Suppose that* $p(e) \neq (x_u^{-1} r(e) x_v)^d$ *for at least one edge* e *and* $w(e) \neq (x_u^{-1} r(e) x_v)^d$ *for at least one edge* e.

(i) *For a fixed* s*,* $\beta(q,s)$ *is a strictly convex function of* q*, and for a fixed* q*,* $\beta(q,s)$ *is a strictly convex function of* s.

(ii) *For a fixed* s*,* $\alpha_p(q,s)$ *is a strictly decreasing funtion of* q*, and for a fixed* q*,* $\alpha_w(q,s)$ *is a strictly decreasing funtion of* s.

(iii) $K_u^{(\alpha_p,*)} = \emptyset$ *if* $\alpha_p < \eta_p^{min}$ *or* $\alpha_p > \eta_p^{max}$. $K_u^{(*,\alpha_w)} = \emptyset$ *if* $\alpha_w < \eta_w^{min}$ *or* $\alpha_w > \eta_w^{max}$.
And so $K_u^{(\alpha_p,\alpha_w)} = \emptyset$ *if* $\alpha_p < \eta_p^{min}$ *or* $\alpha_p > \eta_p^{max}$ *or* $\alpha_w < \eta_w^{min}$ *or* $\alpha_w > \eta_w^{max}$.

The proof is in Appendix.

3. Proof of the dimension theorem

We follow the proof of Falconer([2] p.192).

First we state Proposition 2.3 in [2] for the proof of Theorem 1.

Proposition 3.1 (Falconer, "Fractal Geometry" [2], Proposition 2.3)**.**
Let E *be a Borel set.*

$$\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = f$$

if there exists a finite measure ν *such that* $\nu(E) > 0$ *and for all* $x \in E$*,*

$$\lim_{r\to 0} \frac{\log \nu(B_r(x))}{\log r} = f,$$

where $B_r(x)$ *is the closed ball of with centre* x *and radius* $r > 0$.

Let

$$\Psi_u(q,s,\beta) = \xi_u(q,s)^{-1} \sum_{v\in V} \sum_{e\in E_{uv}} p(e)^q w(e)^s r(e)^{\beta} \xi_v(q,s),$$

where $\{\xi_v(q,s)\}_{v\in V}$ is defined in Section 2, that is,

$$\sum_{v\in V} \sum_{e\in E_{uv}} p(e)^q w(e)^s r(e)^{\beta(q,s)} \xi_v(q,s) = \xi_u(q,s) \quad \text{for all} \quad u \in V.$$

Note that

$$\Psi_u(q,s,\beta(q,s)) = 1.$$

Then the following lemma holds.

Lemma 3.1 (Extension of Lemma 11.3 of Falconer [2]).
For all $\epsilon > 0$,

$$\Psi_u(q+\delta, s, \beta(q,s) + (-\alpha_p(q,s)+\epsilon)\delta) < 1,$$
$$\Psi_u(q-\delta, s, \beta(q,s) + (\alpha_p(q,s)+\epsilon)\delta) < 1,$$
$$\Psi_u(q, s+\delta, \beta(q,s) + (-\alpha_w(q,s)+\epsilon)\delta) < 1$$

and

$$\Psi_u(q, s-\delta, \beta(q,s) + (\alpha_w(q,s)+\epsilon)\delta) < 1$$

for all sufficiently small $\delta > 0$.

The proof is given in Appendix.

We define a probability measure $\hat{\nu}_u^{q,s}$ on $\hat{K}_u$ $(u \in V)$ by

$$\hat{\nu}_u^{q,s}([\gamma]) = \xi_u(q,s)^{-1} p(\gamma)^q w(\gamma)^s r(\gamma)^{\beta(q,s)} \xi_v(q,s) \quad \text{for} \quad \gamma \in E_{uv}^k.$$

Note that $\sum_{\gamma \in E_u^k} \hat{\nu}_u^{q,s}([\gamma]) = 1$. The corresponding measure $\nu_u^{q,s}$ on K_u $(u \in V)$ is defined by $\nu_u^{q,s}(F) = \hat{\nu}_u^{q,s}(h_u^{-1}(F))$.

Recall that

$$\alpha_p(q,s) = -\frac{\partial\beta(q,s)}{\partial q}, \quad \alpha_w(q,s) = -\frac{\partial\beta(q,s)}{\partial s},$$

and

$$f(q,s) = \alpha_p(q,s)q + \alpha_w(q,s)s + \beta(q,s).$$

We have the following result.

Proposition 3.2 (Extension of Proposition 11.4 of Falconer [2]).
(a) $\nu_u^{q,s}(K_u^{(\alpha_p(q,w),\alpha_w(q,s))}) = 1$ *for* $u \in V$.
(b) *For all* $x \in K_u^{(\alpha_p,\alpha_w)}$ *where* $\alpha_p = \alpha_p(q,w)$ *and* $\alpha_w = \alpha_w(q,s)$ *for some* q *and* s, *we have* $\log \nu_u^{q,s}(\mathrm{B}(x,r))/\log r \to f(q,s)$ *as* $r \to 0$.

Proof

(a) Let $\epsilon > 0$ be given. Let $\delta > 0$ be sufficiently small in Lemma 3.1. Let $x \in K_u$ and $X_k(x) = \theta_{e_1}\theta_{e_2}\cdots\theta_{e_k}(J_v)$ where $e_1e_2\cdots e_k \in E_{uv}^{(k)}$ and $x \in \theta_{e_1}\theta_{e_2}\cdots\theta_{e_k}(J_v)$.

Let $a_1 = \max\{\xi_v(q+\delta,s)\xi_v(q,s)^{-1} : v \in V\}, a_2 = \max\{\xi_v(q+\delta,s)^{-1}\xi_v(q,s) : v \in V\}$ and $c = \max\{\Psi_v(q+\delta, s, \beta(q,s) + (\epsilon - \alpha_p(q,s))\delta) : v \in V\}$. By Lemma 3.1, we have $0 < c < 1$. Let $|X| = \sup\{|x-y| : x, y \in X\}$.

Then we have

$$
\begin{aligned}
&\nu_u^{q,s}(x : \rho_u(X_k(x)) \geq |X_k(x)|^{\alpha_p-\epsilon}) \\
&\quad = \nu_u^{q,s}(x : 1 \leq \rho_u(X_k(x))^{\delta}|X_k(x)|^{(\epsilon-\alpha_p)\delta}) \\
&\quad \leq \int \rho_u(X_k(x))^{\delta}|X_k(x)|^{(\epsilon-\alpha_p)\delta} d\nu_u^{q,s}(x) \\
&\quad = \xi_u(q,s)^{-1} \sum_{v\in V} \sum_{\gamma\in E_{uv}^{(k)}} p(\gamma)^{\delta} r(\gamma)^{(\epsilon-\alpha_p)\delta} p(\gamma)^q w(\gamma)^s r(\gamma)^{\beta(q,s)} \xi_v(q,s) \\
&\quad = \xi_u(q,s)^{-1} \sum_{v\in V} \sum_{\gamma\in E_{uv}^{(k)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_v(q,s) \\
&\quad = \xi_u(q,s)^{-1} \sum_{y\in V} \sum_{\gamma\in E_{uy}^{(k-1)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_y(q+\delta,s) \\
&\xi_y(q+\delta,s)^{-1} \sum_{v\in V} \sum_{e\in E_{yv}} p(e)^{q+\delta} w(e)^s r(e)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_v(q+\delta,s) \\
&\xi_v(q+\delta,s)^{-1} \xi_v(q,s) \\
&\quad \leq \xi_u(q,s)^{-1} \sum_{y\in V} \sum_{\gamma\in E_{uy}^{(k-1)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_y(q+\delta,s) \\
&\Psi_y(q+\delta, s, \beta(q,s) + (\epsilon-\alpha_p)\delta)\ \xi_v(q+\delta,s)^{-1} \xi_v(q,s) \\
&\quad \leq \xi_u(q,s)^{-1} \sum_{y\in V} \sum_{\gamma\in E_{uy}^{(k-1)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_y(q+\delta,s)\ c\ a_2 \\
&\quad = \xi_u(q,s)^{-1} \sum_{z\in V} \sum_{\gamma\in E_{uz}^{(k-2)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_z(q+\delta,s) \\
&\xi_z(q+\delta,s)^{-1} \sum_{y\in V} \sum_{e\in E_{zy}} p(e)^{q+\delta} w(e)^s r(e)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_y(q+\delta,s)\ c\ a_2 \\
&\quad \leq \xi_u(q,s)^{-1} \sum_{z\in V} \sum_{\gamma\in E_{uz}^{(k-2)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_z(q+\delta,s) \\
&\Psi_z(q+\delta, s, \beta(q,s) + (\epsilon-\alpha_p)\delta)\ c\ a_2 \\
&\quad \leq \xi_u(q,s)^{-1} \sum_{z\in V} \sum_{\gamma\in E_{uz}^{(k-2)}} p(\gamma)^{q+\delta} w(\gamma)^s r(\gamma)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_z(q+\delta,s)\ c^2\ a_2 \\
&\cdots \\
&\quad \leq \xi_u(q,s)^{-1} \xi_u(q+\delta,s) \\
&\xi_u(q+\delta,s)^{-1} \sum_{x\in V} \sum_{e\in E_{ux}} p(e)^{q+\delta} w(e)^s r(e)^{\beta(q,s)+(\epsilon-\alpha_p)\delta} \xi_x(q+\delta,s)\ c^{k-1}\ a_2 \\
&\quad \leq \xi_u(q,s)^{-1} \xi_u(q+\delta,s) \Psi_u(q+\delta, s, \beta(q,s) + (\epsilon-\alpha_p)\delta)\ c^{k-1}\ a_2 \\
&\quad \leq \xi_u(q,s)^{-1} \xi_u(q+\delta,s)\ c^k\ a_2 \leq a_1 a_2 c^k.
\end{aligned}
$$

We have

$$\nu_u^{q,s}(x : \rho_u(X_k(x)) \geq |X_k(x)|^{\alpha_p - \epsilon} \quad \text{for some} \quad k \geq k_0) \leq \sum_{k=k_0}^{\infty} a_1 a_2 c^k < a_1 a_2 c^{k_0}/(1-c).$$

It follows that for $\nu^{(q,s)}$-almost all x,

$$\lim_{k\to\infty} \inf \log \rho_u(X_k(x))/ \log |X_k(x)| \geq \alpha_p - \epsilon.$$

This holds for all $\epsilon > 0$, and so

$$\alpha_p \leq \lim_{k\to\infty} \inf \log \rho_u(X_k(x))/ \log |X_k(x)|.$$

Similarly we have for suffiently small $\delta > 0$,

$$\nu_u^{q,s}(x : \rho_u(X_k(x)) \leq |X_k(x)|^{\alpha_p + \epsilon} \quad \text{for some} \quad k \geq k_0) \leq \sum_{k=k_0}^{\infty} b_1 b_2 d^k < b_1 b_2 d^{k_0}/(1-d)$$

where $b_1 = \max\{\xi_v(q-\delta, s)\xi_v(q, s)^{-1} : v \in V\}, b_2 = \max\{\xi_v(q-\delta, s)^{-1}\xi_v(q, s) : v \in V\}$ and $d = \max\{\Psi_v(q-\delta, s, \beta(q, s) + (\epsilon + \alpha_p)\delta) : v \in V\}$ and $0 < d < 1$.

So

$$\lim_{k\to\infty} \sup \log \rho_u(X_k(x))/ \log |X_k(x)| \leq \alpha_p.$$

It means that for $\nu_u^{q,s}$-a.e. $x \in K_u$

$$\lim_{k\to\infty} \log \rho_u(X_k(x))/ \log |X_k(x)| = \alpha_p.$$

It holds that by Edgar-Mauldin([1] p.610),

$$\lim_{r\to 0} \frac{\log \rho_u(B(x, r))}{\log r} = \alpha_p \quad \text{if and only if} \quad \lim_{k\to\infty} \frac{\log \rho_u(X_k(x))}{\log |X_k(x)|} = \alpha_p.$$

So for $\nu_u^{q,s}$-almost all x,

$$\lim_{r\to 0} \frac{\log \rho_u(B_r(x))}{\log r} = \lim_{k\to\infty} \frac{\log \rho_u(X_k(x))}{\log |X_k(x)|} = \alpha_p.$$

We have $\nu_u^{q,s}(K_u^{(\alpha_p(q,s),\ *)}) = 1$.

In the same way we obtain for $\nu_u^{q,s}$-a.e. $x \in K_u$

$$\lim_{k\to\infty} \log \varrho_u(X_k(x))/ \log |X_k(x)| = \alpha_w,$$

and for $\nu_u^{q,s}$-almost all $x \in K_u$,

$$\lim_{r\to 0} \frac{\log \varrho_u(B(x, r))}{\log r} = \lim_{k\to\infty} \frac{\log \varrho_u(X_k(x))}{\log |X_k(x)|} = \alpha_w,$$

and $\nu_u^{q,s}(K_u^{(*,\alpha_w(q,s))}) = 1$.

Since $K_u^{(\alpha_p(q,s),\alpha_w(q,s))} = K_u^{(\alpha_p(q,s),*)} \cap K_u^{(*,\alpha_w(q,s))}$,

$$\nu_u^{q,s}(K_u^{(\alpha_p(q,s),\alpha_w(q,s))}) = 1.$$

(b) We have

$$\frac{\log \nu_u^{q,s}(X_k(x))}{\log |X_k(x)|} = \frac{\log \xi_u(q,s)^{-1}}{\log |X_k(x)|} + q\,\frac{\log \rho_u(X_k(x))}{\log |X_k(x)|} + s\,\frac{\log \varrho_u(X_k(x))}{\log |X_k(x)|} + \beta(q,s)\frac{\log |X_k(x)|}{\log |X_k(x)|} + \frac{\log \xi_v(q,s)}{\log |X_k(x)|}.$$

For all $x \in K^{(\alpha_p(q,s),\alpha_w(q,s))}$,

$$\lim_{k\to\infty} \frac{\log \nu_u^{q,s}(X_k(x))}{\log |X_k(x)|} = q\,\alpha_p(q,s) + s\,\alpha_w(q,s) + \beta(q,s) = f(q,s).$$

As noted before

$$\lim_{r\to 0} \frac{\log \nu_u^{q,s}(B(x,r))}{\log r} = \lim_{k\to\infty} \frac{\log \nu_u^{q,s}(X_k(x))}{\log |X_k(x)|},$$

and so we have the result.

Proof of Theorem 1

The facts that

$$\dim_{\mathrm{H}} K_u^{(\alpha_p,*)} = \dim_{\mathrm{P}} K_u^{(\alpha_p,*)} = f(q,0)$$

where $\alpha_p = -\frac{\partial\beta(q,0)}{\partial q}$ for some q, and that

$$\dim_{\mathrm{H}} K_u^{(*,\alpha_w)} = \dim_{\mathrm{P}} K_u^{(*,\alpha_w)} = f(0,s)$$

where $\alpha_w = -\frac{\partial\beta(0,s)}{\partial s}$ for some s are proved in Edgar-Mauldin [1].

By Proposition 3.2, $\nu_u^{q,s}$ is concentrated on $K_u^{(\alpha_p,\alpha_w)}$ with $\alpha_p = -\frac{\partial\beta}{\partial q}(q,s)$ and $\alpha_w = -\frac{\partial\beta}{\partial s}(q,s)$. For $x \in K_u^{(\alpha_p,\alpha_w)}$, we have $\nu_u^{q,s}(B_r(x))/\log r \to f(q,s)$ as $r \to 0$.

By Proposition 3.1, we have the result.

Appendix

Proof of Proposition 2.3

Proof of (A)

(i) Let q and s be given. Write $\xi_v = x_v^{d-dq-ds}$ where $x_v > 0$ and

$$\sum_{v\in V}\sum_{e\in E_{uv}} r(e)^d x_v^d = x_u^d \quad \text{for all} \quad u \in V.$$

Then

$$\begin{aligned}\sum_{v}\sum_{e\in E_{uv}} p(e)^q w(e)^s r(e)^{d-dq-ds}\xi_v &= \sum_{v}\sum_{e\in E_{uv}} (x_u^{-1} r(e)x_v)^{dq}(x_u^{-1} r(e)x_v)^{ds} r(e)^{d-dq-ds} x_v^{d-dq-ds} \\ &= x_u^{-dq-ds}\sum_{v} -dq-ds \sum_{v}\sum_{e\in E_{uv}} r(e)^d x_v^d \\ &= x_u^{-dq-ds} x_u^d = \xi_u.\end{aligned}$$

Therefore $\Phi(q,s,d-dq-ds) = 1$, and so $\beta(q,s) = d-dq-ds$.

(ii) By (i), $\alpha_p(q,s) = -\frac{\partial\beta}{\partial q}(q,s) = d$ and $\alpha_w(q,s) = -\frac{\partial\beta}{\partial s}(q,s) = d$.

(iii) By (i), $f = q\alpha_p + s\alpha_w + \beta = dq + ds + (d - dq - ds) = d$.

(iv) Let $\gamma = e_1 e_2 \dots e_k \in E_{uv}^{(k)}$, and $x_{min} = \min_v x_v$ and $x_{max} = \max_v x_v$.

Note that

$$p(\gamma) = \prod_{i=1}^{k} p(e_i) = \prod_{i=1}^{k} (x_{u_i}^{-1} r(e) x_{v_i})^d = (x_u^{-1} r(\gamma) x_v)^d$$

where $e_i \in E_{u_i v_i}$.

So

$$\frac{\log p(\gamma)}{\log r(\gamma)} = d\left(1 + \frac{\log(x_v/x_u)}{\log r(\gamma)}\right).$$

Let $\sigma \in E_u^{(k)}$, then

$$\frac{\log p(\sigma|k)}{\log r(\sigma|k)} = d\left(1 + \frac{\log(x_v/x_u)}{\log r(\sigma|k)}\right),$$

where $\sigma|k \in E_{uv}^{(k)}$.

Therefore $\log p(\sigma|k)/\log r(\sigma|k) \to d$ as $k \to \infty$, because $\log r(\sigma|k) \to -\infty$ and $x_{min}/x_{max} \le x_v/x_u \le x_{max}/x_{min}$.

Similarly we have $\log w(\sigma|k)/\log r(\sigma|k) \to d$ as $k \to \infty$.

Proof of (B) and (B')

(i) Let $\phi_w(s) = \beta(0,s)$, i.e., $\Phi(0, s, \phi_w(s)) = 1$. Let $y_v(s)$ be the right eigen vector for which

$$\sum_{v\in V} \sum_{e\in E_{uv}} w(e)^s r(e)^{\phi_w(s)} y_v(s) = y_u(s)$$

for all $u \in V$.

Let $\xi_v(s) = y_v(s) x_v^{-dq}$, then

$$\begin{aligned}\sum_{v\in V} \sum_{e\in E_{uv}} p(e)^q w(e)^s r(e)^{-dq+\phi_w(s)} \xi_v(s) &= \sum_{v\in V} \sum_{e\in E_{uv}} (x_u^{-1} r(e) x_v)^{dq} w(e)^s r(e)^{-dq+\phi_w(s)} y_v(s) x_v^{-dq} \\ &= x_u^{-dq} \sum_{v\in V} \sum_{e\in E_{uv}} w(e)^s r(e)^{\phi_w(s)} y_v(s) = x_u^{-dq} y_u(s) \\ &= \xi_u(s).\end{aligned}$$

Therefore $\Phi(q, s, -dq + \phi_w(s)) = 1$, so $\beta(q,s) = -dq + \phi_w(s)$.

The fact that $\beta(q,s)(= dq + \phi_w(s))$ is a strictly convex function of s, i.e., $\phi_w(s)$ is a strictly convex function of s is proved in the same way as the proof of Proposition 3.3 (B) in Edgar-Mauldin([1]).

(ii),(iii) and (iv) are also proved in the same way as the proof of Proposition 3.3 (B) in Edgar-Mauldin([1]).

Proof of (B') is similar.

Proof of (C)

The proof is similar to the proof of Proposition 3.3 (B) of Edgar and Mauldin ([1]).

Proof of Lemma 3.1

Note that

$$1 = \Psi_u(q+\delta, s, \beta(q+\delta, s)) = \xi_u(q+\delta, s)^{-1} \sum_{v \in V} \sum_{e \in E_{uv}} p(e)^{q+\delta} w(e)^s r(e)^{\beta(q+\delta,s)} \xi_v(q+\delta, s).$$

Recalling that $\frac{\partial \beta}{\partial q} = -\alpha$, we have

$$\beta(q+\delta, s) = \beta(q, s) - \alpha(q, s)\,\delta + \mathrm{O}(\delta^2) < \beta(q, s) + (-\alpha(q, s) + \epsilon)\delta,$$

for sufficietly small $\delta > 0$. Since $0 < r(e) < 1$, we have

$$1 = \Psi_u(q+\delta, s, \beta(q+\delta, s)) > \Psi_u(q+\delta, s, \beta(q, s) + (-\alpha(q, s) + \epsilon)\delta).$$

Similarly we have other inequalities.

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相互帰納的集合の確率と重みに対応するマルチフラクタル分解

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要 旨

相互帰納的集合（graph-directed 集合）を対象とし，その上の確率と重みを扱う．確率密度と重み密度のペアに対するマルチフラクタル分解を調べる．そのためにパラメーターの対 $(q,\ s)$ を導入する．これらのパラメーターを使って,相互帰納的集合上で，確率密度と重み密度が一定の値をとる部分集合のハウスドルフ次元とパッキング次元を特定する．この拡張は文献 [4] で示されるように,マルチフラクタル構造のスピングラス現象を調べるための自由度を与える．この論文は文献 [1], [3], [4] の拡張である．

キーワード：ランダム反復アルゴリズム，相互帰納的集合，マルチフラクタル分解，ハウスドルフ次元，確率と重み