連続体上の計算可能性－極限計算の諸理論の相互関係

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研究目的

離散構造、実数および実数（連続体）上の連続関数の「計算可能性」の概念は、現在までにほぼ確立している。しかし科学技術計算では、ある種の不連続関数が重要であり、その計算可能性概念の確立は計算論の重要な課題である。近年関連する多くの理論が提案されてきた。当研究組織の構成員は、とくに区分的連続関数（孤立点以外で連続な関数）の計算可能性理論として、主に極限計算理論、一様位相空間論、Fine 距離理論、ドメイン理論の手法を提案し、共同研究を続け、それぞれが有効な手法であることを実証してきた。今年度の申請課題の目的は、極限計算理論を基礎に、上記の手法の間のある種の同値性を示し、科学技術計算で重要な関数族の計算可能性概念に関するパラダイムの確立を目指すことにある。

プロジェクトによる主な成果

［1］ Mariko Yasugi: “Computability problems on the continuum” Proceedings of “The Symposium on Mathematical Logic ’03” in Kobe, 2003, 11 pages.

［2］ Mariko Yasugi, Yoshiki Tsujii, Takakazu Mori: “Effective sequence of uniformities and the Rademacher function system” available at http://www.kyoto-su.ac.jp/~yasugi/page/recent.html

以下に［1］と［2］にしたがって研究の背景および成果の概説をする。

Computability on the Continuum

1 Introduction

It is important and mathematically significant to review some theories of mathematics from an algorithmic standpoint.

In studies of algorithm in analysis, one puts the basis of considerations on the computability of real numbers and the computability of continuous functions.

Here a real number x is said to be computable if there is a sequence of rational numbers (fractions) $\{r_n\}$ which approximates x and satisfies the following two conditions.

(1) The fractional sequence $\{r_n\}$ is recursive.

(2) There is a recursive modulus of convergence (approximation).

When the condition (2) holds, we say that x is effectively approximated by $\{r_n\}$, or $\{r_n\}$ effectively converges to x. In general, we use the expression effective when a condition similar to (2) is satisfied.

A computable sequence of real numbers can also be defined in a similar manner. One needs the computability of a sequence of real numbers when one has to refer to the limit.

The family of all computable sequences of real numbers is called the computability structure of the field of real numbers.

The computability of a continuous real function on a compact interval with computable end points can be defined in a natural manner. A real function f (on a compact interval) is computable if the following hold.

(3) f preserves sequential computability, that is, for any input of a computable sequence of real numbers, its output by f is also a computable sequence.

(4) f has a recursive modulus of uniform continuity.

Computability on an open interval can be defined in terms of an approximation of the interval by a sequence of compact intervals and a modulus of uniform continuity which is recursive relative to the approximating intervals.

These notions of computability respectively of a real number (a sequence of real numbers) and of a continuous function (a sequence of continuous functions) are generally agreed to be natural and in a sense the strongest.

According to the definition described in (3) and (4) above for a continuous function, computability means that there is a way to nicely approximate the values for computable inputs, and this notion depends on the continuity.

Very often, however, we compute values and draw a graph of a discontinuous function. We can, let Mathematica, for example, draw graphs of some discontinuous functions. We thus expect that some class of discontinuous functions can be attributed a certain kind of computability. In an attempt of computing a discontinuous function, a problem arises in the computation of the value at a jump point (a point of discontinuity). This is because it is not in general decidable if a real number is a jump point, that is, the question "$x=a$?" is not decidable even for computable x and a.

(For the subsequent discussion, let us here note the following: =, ≤, < on natural numbers and fractional numbers are decidable. $a<b$ is decidable for computable real numbers a and b, while $a=b$ and $a\leq b$ are not necessarily decidable even for computable real numbers.)

One method of dissolving this problem was proposed in [4] by Pour-El and Richards. It was a functional analysis approach, that is, a function is regarded as computable if it can be effectively approximated by effectively enumerated rational coefficient polynomials with respect to the norm of a function space, such as a Banach space or a Fréchet space.

In such a case, a function is regarded as computable as a point in a space. This is sufficient in order to draw a rough graph of the function, but does not supply us with information when computing individual values.

There are many ways of characterizing computation of a discontinuous function. Here we will report some of the approaches to this problem which we have undertaken so far[1]. One is to express the value of a function at a jump point in terms of limiting recursive funcions instead of recursive functions ([7]). Another is to change the topology of the domain of a function ([6]). In fact they are equivalent ([8]).

Pour-El theory as well as its succeeding works on computability structures for Fréchet spaces and metric spaces are also explained in [10].

2 Preliminaries

The basic definitions below are taken from [4]. A sequence of ratinal numbers $\{r_n\}$ is called *recursive* if

$$r_n=(-1)^{\beta(n)}\frac{\gamma(n)}{\delta(n)}$$

with recursive β, γ and δ.

A real number x is called *computable* (**R**-computable) if

$$\forall m\geq\alpha(p)\cdot|x-r_m|<\frac{1}{2^p}$$

for recursive α and $\{r_m\}$. We will express such a circumstance as $x\cong\langle r_m,\ \alpha\rangle$.

These definitions can be extended to a *computable sequence* of real numbers.

[1]This work has been supported in part by JSPS 12440031 as well as by KSU-Project03.

A real (continuous) function f is *computable* ($\mathbf{R}$-computable) if the following hold.

(i) f preserves *sequential computability*, that is, for a computable $\{x_n\}$, $\{f(x_n)\}$ is computable.

(ii) f is continuous with *recursive modulus of continuity*, say β;

$$\forall p \forall n \in \mathbf{N}^{+} \forall k \geq \beta(n, p) \forall x, y \in [n, n+1].$$
$$|x-y| < \frac{1}{2^k} \Rightarrow |f(x)-f(y)| < \frac{1}{2^p}.$$

This can be extended to a computable sequence of functions.

3 Computation in the limit

As a start, we will try to compute $g(x)=\frac{1}{2}[x]$, where $[x]$ is the Gaussian function, according to [7].

Let x be a computable real number with $x \cong \langle r_m, \alpha \rangle$, and let us consider how to compute the value $g(x)$. For the sake of simplicity, we assume $x>0$. From the information on x, one can effectively find an n such that $n<x<n+2$. Then check

$$r_{\alpha(p)} < (n+1)-1/2^p?$$

According to the answer to this inquiery, we define a sequence of integers $\{N_p\}$ as follows. While the answer is *No*, put $N_p=n+1$. Once the answer becomes *Yes* at p, then put $N_q=n$ for all q satisfying $q \geq p$. The sequence $\{N_p\}$ is well-defined and recursive.

Define next a recursive sequence of rational numbers $r_p=\frac{N_p}{2}$. If $N_p=n+1$ holds for all p, then the limit of the sequence $\{r_p\}$ is $\frac{n+1}{2}$; otherwise, the limit is $\frac{n}{2}$. In either case, the sequence $\{r_p\}$ approximates the value $\frac{1}{2}[x]$. For each case, there is a recursive modulus of continuity; only, we cannot decide which is the case.

This undecidability indicates that, although there is a computation algorithm for each x, it does not guarantee a master program to compute the value $\frac{1}{2}[x]$. Indeed, there is a computable sequence of real numbers $\{x_n\}$ for which the sequence of values $\left\{\frac{1}{2}[x_n]\right\}$ is not computable. On the other hand, if we allow a limiting recursive function for a modulus of convergence, then we can claim the following: for any computable sequence of real numbers $\{x_m\}$, there is a recursive sequence of rational numbers $\{q_{mi}\}$ which approximates $\left\{\frac{1}{2}[x_m]\right\}$ with a modulus of convergence which is limiting recursive.

The limiting recursive function is defined as follows.

Definition 3.1 (Limiting recursive function: Gold [1]) Let r, $s \geq 0$ be integers and let g and ϕ_1, $\cdots$, ϕ_r be recursive functions. The partial function h defined as follows will be called *limiting recursive:*

$$h(p_1, \cdots, p_s) = \lim_n g(\tilde{\phi}_1(n), \cdots, \tilde{\phi}_r(n), p_1, \cdots, p_s, n),$$

where $\tilde{\phi}(n)$ is a code for the finite sequence

$$\langle \phi(0, p_1, \cdots, p_s), \cdots, \phi(n, p_1, \cdots, p_s) \rangle,$$

Examples

$$h(p_1, \cdots, p_s) = \lim_n \phi(n, p_1, \cdots, p_s).$$

$h(p_1, \cdots, p_s)$ = the least value of $\phi(n, p_1, \cdots, p_s)$ with respect to n.

There are many examples of real functions which can be computed using the limiting recursive modulus of convergence: see examples below. They are all piecewise continuous functions, jumping at some computable points. It is hence sensible to confine ourselves to such functions as a start of studying computability problems of discontinuous functions.

Examples ([7], [11]) $h(x) = x - [x]$;$]x[= n$ if and only if $n < x \leq n+1$; $\sigma(x) = 1\,(x \in (0, \infty))$, $= \frac{1}{2}\,(x=0)$, $= 0\,(x \in (-\infty, 0))$; the Rademacher function system; $\tau(x) = \tan x$ if $\frac{2n+1}{2}\pi < x < \frac{2n+3}{2}\pi$ and $\tau(x) = 0$ if $x = \frac{2n+1}{2}\pi$.

4 Topological computability

In computing the values or drawing the graph of a piecewise continuous function, it is a usual practice to first compute the value or plot a dot at a jump point, and then compute values or draw a curve on the open interval where the function is continuous. Such an action corresponds to the mathematical notion of isolating the jump points. We are thus led to the uniform topology of the real line induced from the Euclidean topology by isolating the jump points.

Let X be a non-empty set.

A sequence $\{V_n\}_{n\in\mathbf{N}}$ such that $V_n : X \to P(X)$ is called a *uniformity* if it satisfies some axioms, say, Axioms $A_1 \sim A_5$ (to be stated below). In particular, A_1 and A_2 can be unified to

$$\cap_n V_n(x) = \{x\}.$$

We will state Axioms $A_3 \sim A_5$ in the form of effective uniformity. $\Gamma = \langle X, \{V_n\}\rangle$ forms a uniform topological space. Subsequent definitions are due to [6].

Definition 4.1 (Effective uniformity) A uniformity $\{V_n\}$ on X is *effective* if there are recursive functions α_1, α_2, α_3 which satisfy the following.

$$\forall n,\ m\in\mathbf{N}\,\forall x\in X,\ V_{\alpha_1(n,m)}(x) \subset V_n(x) \cap V_m(x) \quad \text{(effective } A_3\text{)};$$

$$\forall n\in\mathbf{N}\,\forall x,\ y\in X,\ x\in V_{\alpha_2(n)}(y) \to y\in V_n(x) \quad \text{(effective } A_4\text{)};$$

$$\forall n\in\mathbf{N}\,\forall x,\ y,\ z\in X,\ x\in V_{\alpha_3(n)}(y),\ y\in V_{\alpha_3(n)}(z) \to x\in V_n(z) \quad \text{(effective } A_5\text{)}.$$

Definition 4.2 (Effective convergence) $\{x_k\} \subset X$ *effectively converges* to x in X if there is a recursive function γ satisfying $\forall x \forall k \geq \gamma(n)(x_k \in V_n(x))$.

This can be extended to effective convergence of a multiple sequence.

Definition 4.3 (Computability structure) Let S be a family of sequences from X (multiple sequences included). S is called a *computability structure* if the following hold.

C1: (Non-emptiness) S is nonempty.

C2: (Re-enumeration) If $\{x_k\} \in S$ and α is a recursive function, then $\{x_{\alpha(i)}\}_i \in S$.

This can be extended to multiple sequences.

C3: (Limit) If $\{x_{lk}\}$ belongs to S, $\{x_l\}$ is a sequence from X, and $\{x_{lk}\}$ converges to $\{x_l\}$ effectively, then $\{x_l\} \in S$. (S is closed with respect to effective convergence.)

This can be extended to multiple sequences.

A sequence belonging to S is called *computable*, and x is *computable* if $\{x, x, \cdots\}$ is computable.

We will henceforth consider the space

$$C_T = \langle X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3, S\rangle.$$

Definition 4.4 (Effective approximation) $\{e_k\} \in S$ is an *effective approximating set* of S: $\forall \{x_l\}$ computable, there is a recursive function ν such that

$$\forall n \forall l(e_{\nu(n,l)} \in V_n(x_l)).$$

Definition 4.5 (Effective separability) Suppose $\{e_k\}$ is an effective approximating set and dense in X:

$$\forall n \forall x \exists k(e_k \in V_n(x)).$$

Then C_T is *effectively separable*, and $\{e_k\}$ is called an *effective separating set.*

Note Classically, a general method to define a metric d* from a countable uniformity is known. It is an open problem if this induced metric preserves computability. We can, however, show that, effective convergence respectively with respect to an effective uniformity and with respect to the induced metric are equivalent ([9]).

Definition 4.6 (Relative computability) (1) $f: X \to \mathbf{R}$ is *relatively computable* (with respect to S) if:

(i) f preserves sequential computability, that is, if $\{x_m\}$ is C_T-computable, then $\{f(x_m)\}$ is an $\mathbf{R}$-computable sequence of real numbers.

(ii) For any $\{x_m\} \in S$ there exists a recursive function $\gamma(m, p)$ such that $y \in V_{\gamma(m,p)}(x_m)$ implies $|f(y) - f(x_m)| \leq \frac{1}{2_p}$.

(2) (1) can be extended to a sequence of functions.

Definition 4.7 (Computable function) (1) $f: X \to \mathbf{R}$ is *computable* if the following hold.

(i) f preserves sequential computability.

(ii) f is relatively computable, and there exist an effective approximating set, say $\{e_k\} \in S$, and a recursive function $\gamma_0(k,p)$ for which

$$y \in V_{\gamma_0(k,p)}(e_k) \text{ implies } |f(y) - f(e_k)| \leq \frac{1}{2^p}$$

and

$$\bigcup_{k=1}^{\infty} V_{\gamma_0(k,p)}(e_k) = X$$

for p.

(2) (1) can be extended to a sequence of functions.

Definition 4.8 (Uniform computability) f is *uniformly computable* if f preserves sequential computability and there is a recursive modulus of uniform continuity for f.

There are many interesting examples of the effective uniform space and computable/uniformly computable functions on such a space (cf. [6], [8]). For some of them, equivalence of sequential computability with respect to the uniform topology and limiting sequential computability with respect to the Euclidean topology has been demonstrated (cf. [8]).

5 A sequence of uniform spaces

We now confine ourselves to the real numbers in the interval $I=[0, 1]$ and functions on it. The theory of an effective sequence of uniformities on I and its limit is developed in [b]. The "limit uniformity" is proven to be effectively equivalent to the "diagonal uniformity" and two notions of computability, "diagonal computability" and "ω-computability," are shown to be equivalent. We next proposed the notion of "uniform D-computability" of a piecewise continuous function in the space of the diagonal uniform space and then proved that the Rademacher function system is "uniformly D-computable"

We assume that ν, $k \in \mathbf{N}$ and $0 \leq k \leq 2^{\nu}-1$. We will consider real numbers and sequences of real numbers in I.

Definition 5.1 (Intervals and uniformity) Define subintervals of I, I_k^{ν}, and a sequence of maps $U_n^{\nu}: I \to P(I)$ as follows.

$$I_k^{\nu}=\left[\frac{k}{2^{\nu}}, \frac{k+1}{2^{\nu}}\right)$$

$$U_n^{\nu}(x)=I_k^{\nu} \cap \left(x-\frac{1}{2^n}, x+\frac{1}{2^n}\right) \text{ if } x \in I_k^{\nu}.$$

Note 1) $\{U_n^{\nu}\}$ forms an "effective sequence of uniformities" on I.

2) Put $Z_{(\nu,n)}=U_n^{\nu}$. $\{Z_{(\nu,n)}\}$ is the "effective limit" of $\{U_n^{\nu}\}$.

Definition 5.2 (ν-computability) Let ν be an arbitrary (but fixed) natural number.

1) A sequence $\{a_{\mu i}\}$ with multiple index μi is called a ν-sequence if, for a $k=k_{\mu} \leq 2^{\nu}-1$, $\{a_{\mu i}\}_i \subset I_k^{\nu}$. ($\mu$ may be empty.)

2) A multiple sequence of rational numbers $\{r_{\mu i}\}$ is called a *recursive ν-sequence* if it is recursive and is a ν-sequence.

3) $\{a_{\mu m i}\}$ *converges ν-effectively* to $\{x_{\mu m}\}$ with respect to i if there is a recursive α so that $i \geq \alpha(\mu, m, p)$ implies $a_{\mu m i} \in U_p^{\nu}(x_{\mu m})$. We write this property as

$$x_{\mu m} \cong_{\nu} \langle a_{\mu m i},\ \alpha(\mu,\ m,\ p) \rangle$$

or, for short, $x_{\mu m} \cong_{\nu} \langle a_{\mu m i},\ \alpha \rangle$.

4) A sequence of real numbers $\{x_{\mu m}\}$ is called ν-*computable* if there are recursive ν-sequence $\{r_{\mu m i}\}$ and α as in 3), that is,

$$x_{\mu m} \cong_{\nu} \langle r_{\mu m i},\ \alpha(\mu,\ m,\ p) \rangle.$$

Definition 5.3 (ω-computability) 1) A sequence $\{a^{\nu}_{\mu i}\} \subset I$ is called a $\{\nu\}$-*sequence* if, for each ν, for a $k^{\nu}_{\mu} \leq 2^{\nu}-1$, $\{a^{\nu}_{\mu i}\}_i \subset I^{\nu}_{k^{\nu}_{\mu}}$.

2) A multiple sequence of rational numbers from I, say $\{r^{\nu}_{\mu i}\}$, is called a *recursive* $\{\nu\}$-*sequence* if it is recursive and is a $\{\nu\}$-sequence.

In this case, $\{k^{\nu}_{\mu}\}$ can be recursive, for we can find k^{ν}_{μ} by cheking "$r^{\nu}_{\mu 1} \in I^{\nu}_{\mu}$?".

3) A multiple $\{\nu\}$-sequence $\{a^{\nu}_{\mu m i}\}$ *converges* $\{\nu\}$-*effectively* to $x_{\mu m}$ (with respect to i) if there is a recursive α such that

$$\forall \nu \forall \mu \forall m \forall p \forall i \geq \alpha(\nu;\ \mu,\ m,\ p).\ a^{\nu}_{\mu m i} \in U^{\nu}_{p}(x_{\mu m}).$$

This fact will be expressed by

$$x_{\mu m} \cong_{\omega} \langle a^{\nu}_{\mu m i},\ \alpha(\nu;\ \mu,\ m,\ p) \rangle$$

or simply

$$x_{\mu m} \cong_{\omega} \langle a^{\nu}_{\mu m i},\ \alpha \rangle.$$

4) A sequence $\{x_{\mu m}\}$ is ω-*computable* if there are a recursive $\{\nu\}$-sequence $\{r^{\nu}_{\mu m i}\}$ and a recursive α as in 3), that is,

$$x_{\mu m} \cong_{\omega} \langle r^{\nu}_{\mu m p},\ \alpha(\nu;\ \mu,\ m,\ i) \rangle.$$

Proposition 5.1 (**R**-computability, ω-computability and ν-computability) 1) For a single real number $x \in I$, **R**-computability (i. e., computable in the Euclidean topology), ω-computability and ν-computability (for each fixed ν) are equivalent.

2) If $\{x_{\mu m}\}$ is ω-computable, then it is ν-computable for all ν.

3) For each ν, if $\{x_{\mu m}\}$ is ν-computable, then it is **R**-computable, hence by 2), an ω-computable sequence is **R**-computable.

4) For each ν, there is a sequence $\{x_m\}$, which is $\mathbf{R}$-computable but is not ν-computable.

5) For each ν_1, ν_2 where $\nu_2 > \nu_1$, there is a sequence $\{x_m\}$ which is ν_1-computable but not ν_2-computable.

6) If $\nu_1 < \nu_2$ and $\{x_{\mu m}\}$ is ν_2-computable, then it is ν_1-computable.

Proposition 5.2 (ω-computability structure) ω-computable sequences form a computability structure with respect to $\{Z_{\langle \nu, n \rangle}\}$ (cf. Definition 4.3).

Definition 5.4 (Diagonal sequence) The sequence $\{U_n^n\}$ will be called the *diagonal sequence* of $\{U_n^\nu\}$, and will be denoted by $\{U_n\}$.

Proposion 5.3 (Diagonal sequence and limit) The sequence $\{U_n\}$ forms an effective uniformity which is topologically effectively equivalent to the effective limit $\{Z_{\langle \nu, n \rangle}\}$.

Definition 5.5 (Diagonal uniformity) The sequence of diagonals $\{U_n\}$ as in Definition 5.3 will be called the *diagonal uniformity* determined by $\{U_n^\nu\}$ or $\{Z_{\langle \nu, n \rangle}\}$, and the space $\langle I, \{U_n\}\rangle$ will be called the *diagonal space* determined by $\{U_n^\nu\}$.

Definition 5.6 (Diagonal computability) A sequence of real numbers $\{x_m\} \subset I$ is *diagonal computable* if there is a recursive sequence $\{q_{mp}\}$ of rational numbers which converges to $\{x_m\}$ effectively with respect to $\{U_n\}$ in a manner that, for a recursive γ and for $k \geq \gamma(m, p)$, $q_{mk} \in U_p(x_m)$. We will write this property as

$$x_m \cong_D \langle q_{mk}, \gamma \rangle.$$

This definition can be generalized to a multiple sequence.

Proposition 5.4 The family of diagonal computable sequences of real numbers, say R, forms a computability structure for $\langle I, \{U_n\}\rangle$ (cf. C1~C3 of Definition 4.3).

Theorem 1 (Diagonal computability and ω-computability) A sequence $\{x_m\} \subset [0, 1)$ is diagonal computable if and only if it is ω-computable (cf. Definitions 5.6 and 5.3).

Let $\mathbf{U} = \langle X, \{V_n\}, S\rangle$ be an effective uniform space with a computability structure S. We will consider functions from X to $\mathbf{R}$.

Definition 5.7 (Sequential computability) A function sequence $\{f_l\}$ is called sequentially computable in $\mathbf{U}$ if, for every sequence $\{x_m\}$ in S, $\{f_l(x_m)\}$ is an $\mathbf{R}$-computable double sequence of real numbers.

Propsition 5.5 (D-sequential computability of the Rademacher function system) Let $\{\phi_j(x)\}$ be the Rademacher function system. Then it is sequentially computable in the diagonal uniform space. (We will call such a sequence D-sequentially computable.)

The same applies to the Walsh function system.

Definition 5.8 (Effective uniform continuity : Definition 4.5, [6]) A sequence of functions $\{f_n\}$ is called *uniformly computable* in **U** if it is sequentially computable and there is a recursive function α such that

$$y \in V_{\alpha(n,p)}(x) \Rightarrow |f_n(x) - f_n(y)| \leq \frac{1}{2^p}$$

(cf. Definition 4.5 of [6]).

Theorem 2 (Uniform computability of the Rademacher function system) The function sequence $\{\phi_l\}$ is uniformly computable with respect to $\{U_n\}$ (cf. Definition 5.8). We will call this computability "uniformly D-computable."

The same conclusion holds for the Walsh function system.

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